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Andrzej Wiśniewski

WYDAWNICTWO NAUK SPOŁECZNYCH I HUMANISTYCZNYCH UAM

Poznań 2025

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Preface

For years, most of my work was devoted to the logic of questions. In this book, however, issues concerning questions and questioning are carefully skipped. What I am going to present here are ideas and results which, although in some cases elaborated during my work on questions, are of a more general nature.

What is this book about? This book divides into three parts, titled "Generalizing" (chapters 1–3), "Epistemizing" (chapters 4–5), and "Specifying" (chapters 6–11).

The concept of semantic consequence, or briefly *entailment*, is of basic importance to logic. Traditionally, entailment is conceived as a relation between a set of propositions and a proposition, ensuring the transmission (or preservation, if you prefer) of truth between the former to the latter. A known generalization is provided by the concept of multiple-conclusion entailment. Here entailment is construed as a relation between sets of propositions: a set of propositions multiple-conclusion entails a set of propositions just in case the existence of a true proposition in the latter is ensured by the truth of all the propositions in the former. In Chapter 1 of this book I make a step further in generalizing the concept of entailment. A semantic relation between a family of sets of propositions and a set of propositions, dubbed generalized entailment, is defined and examined. The underlying intuition is: if each set in the family contains a true proposition, then the entailed set contains a true proposition. The above condition may seem a weak one, but it gains intuitive plausibility when one thinks of sets of propositions as representing search spaces. Moreover, it can be shown that single- and multiple-conclusion entailments are definable in terms of generalized entailment.

Chapter 2 is devoted to some special case of generalized entailment, dubbed constructive generalized entailment.

The notion of refutation gradually strengthens its position in logic.

Chapter 3 presents a logical calculus which differentiates between proofs of valid formulas, refutations of inconsistent formulas, and refutations of contingent formulas. However, the calculus offers a uniform proof-mechanism for proofs and refutations. This is achieved by the introduction of a kind of conceptual unifier, namely the notion of holistically inconsistent set of formulas. The system "calculates" such sets, or, to be more precise, sequents based on them. Since valid, inconsistent, and contingent formulas correspond to different, yet strictly defined, holistically inconsistent sets, a proof of a sequent based on a set of a given kind can be regarded, depending on a case, as a proof or as a refutation of the corresponding formula.

Although Chapter 1 and Chapter 2 on the one hand, and Chapter 3 on the other address diverse issues, a feature is common to them: in each case an attempt is made to generalize existing accounts. The title of the part of this book comprising chapters 1–3, "Generalizing," reflects this.

The second part of this book, "Epistemizing," contains two chapters which delve into the concept of propositional knowledge and its relatives. Chapter 4 is of a critical nature. It takes under scrutiny two basic approaches to the nature of propositional knowledge, namely the knowledge as true belief "plus something else" account and the knowledge as true conviction account. Three, to my knowledge so far unnoticed, problems/paradoxes concerning knowledge conceived these ways are pointed out and discussed. Chapter 5, in turn, proposes a change in perspective. The concept of being epistemically permitted is introduced. What is permitted is a declarative sentence/proposition, what permits it (or not) is a state intuitively understood a set of alternative accounts of how things are. In general, being epistemically permitted is different from being epistemically possible and from being known; the latter concepts can be defined in terms of the former, however. A relation akin to entailment, dubbed transmission of epistemic permittance, is then defined and analysed.

The remaining chapters of this book constitute its third part, titled "Specifying." In Chapters 6 and 7 two concepts of single- and multiple-conclusion entailment, based on the idea of minimality, are introduced and studied. The analysed entailments, dubbed "strong," are non-Tarskian. In particular, they are not monotone, but, at the same time, they have some intuitively plausible properties which their standard counterparts lack. Chapter 8 is devoted to the issue of emplacement of strong entailments among alternative proposals, including classical, relevant and connexive stances. Chapter 9 presents an application of the concept of strong single-conclusion entailment in the area of belief revision. Some proof-theoretic accounts of strong entailments are offered in Chapter 10. Chapter 11 briefly indicates how the formal apparatus introduced in the previous chapter can be employed in analysing further issues.

Sources. Some, but not all, of the results presented in this book have been already published in my papers.

Chapters 1 and 2 are based on parts of my "Generalized Entailments" [61]. However, sections of [61] devoted to issues of the logic of questions are not included here, and a few additions were made as well.

Chapter 3 is based on my "Towards a uniform account of proofs and refutations" [64] with rewriting.

Chapter 4 presents so far unpublished material.

Chapter 5 is based on parts of "Being permitted, inconsistencies, and question raising" [59] with extensive rewriting and some terminological changes. The sections of [59] devoted to question raising are skipped.

Chapters 6, 7, 8, and 9 are based on parts of my "Entailment, transmission of truth, and minimality" [65] with rewriting. Sections of [65] devoted to argument analysis are omitted. In the case of Chapter 7, section 7.2.4 provides a new development. As for Chapter 9, section 9.3 is new.

Chapter 10 comprises some (rewritten) parts of [65]. Numerous examples were added in order to make the construction presented more comprehensible. Section 10.2 is new.

Section 11.1 of Chapter 11 come from [65], while the remaining sections are new.

An attempt was made to unify terminology and notation. Taking into account the diversity of themes addressed in this book, in order to make its chapters readable separately, each chapter includes either a section which describes the logical apparatus used or a reference to a chapter where this is done. In some cases, however, technical terms (e.g. valuation, sequent, etc.) are construed differently in different chapters, but this is always signalled.

Poznań, July 2025

Part I

Generalizing

Chapter 1

Generalized Entailment

1.1 Generalized Entailment: Intuitions

As for logic, entailment is most often conceived of as a relation between a set of well-formed formulas (wffs for short) on the one hand, and a single wff on the other. Entailment ensures *transmission of truth*: a wff A entailed by a set of wffs X must be true if only all the wffs in X are true. What "must" means here depends on a logic under consideration, and similarly for "truth." The transmission of truth principle falls under the general schema:

(1) *for each* $\mathfrak{M}$*: if all the wffs in* X *are true in* $\mathfrak{M}$*, then* A *is true in* $\mathfrak{M}$

where $\mathfrak{M}$ stands, depending on a case, for: "valuation" (of an appropriate kind), "model", "intended model", "world of a model", and so forth.

Entailment understood in the standard way exhibits a kind of asymmetry: what is entailed is a single wff, while what is entailing it is a set of wffs. If, for some reasons, you prefer symmetry over the lack of it, there are two possible ways of making entailment a relation between sets of wffs. Let X and Y stand for sets of wffs. One may define entailment of Y from X by imposing either of the following conditions:

(2) *for each* $\mathfrak{M}$*: if all the wffs in* X *are true in* $\mathfrak{M}$*, then all the wffs in* Y *are true in* $\mathfrak{M}$*,*

(3) *for each* $\mathfrak{M}$*: if all the wffs in* X *are true in* $\mathfrak{M}$*, then at least one wff in* Y *is true in* $\mathfrak{M}$*.*

The condition (2) leads to a trivial generalization. Obviously, the condition is fulfilled if, and only if X entails every wff in Y. But the case of condition (3) is different. A generalization by the condition (3) gives a well-known concept of *multiple-conclusion entailment*, or mc-entailment for short.[1]

One cannot say that mc-entailment is always definable in terms of entailment. The following observations justify this claim. First, it happens that a set of wffs is mc-entailed, but no wff in the set is entailed. This phenomenon shows up even at the elementary level of Classical Propositional Logic. Here is a simple example. The singleton set $\{p \vee q\}$ mc-entails the set $\{p, q\}$, but neither p nor q is entailed by $\{p \vee q\}$. Second, it is not a general rule that mc-entailment of Y from X reduces to entailment of $\bigvee Y$ (that is, a disjunction of all the wffs in Y) from X. It can happen that Y is an infinite set and the corresponding language lacks infinite disjunctions. More importantly, there are non-classical logics in which mc-entailment of Y from X holds, but entailment of $\bigvee Y$ from X does not hold.

Example 1.1. Let us consider a three-valued propositional logic in which disjunction, $\vee$, is understood according to Table 1.1.[2]

$\vee$	**0**	**i**	**1**
0	**0**	**i**	**1**
i	**i**	**i**	**i**
1	**1**	**1**	**1**

Table 1.1: *McCarthy's disjunction.*

In such a case p does not entail $q \vee p$ because $q \vee p$ can take the value **i** when p takes the (designated) value **1**. On the other hand, the set $\{q, p\}$ is still mc-entailed by the singleton set $\{p\}$.

[1] It is sometimes claimed that the concept of mc-entailment originates from [14] due to introduction of sequents with sequences of wffs in the succedents. The semantic concept of mc-entailment was explicitly introduced in [7] under the heading "involution." Its syntactic counterpart, mc-consequence, was incorporated into the general theory of logical calculi in [39]. The first monograph devoted to mc-consequence and related concepts (multiple-conclusion calculus, multiple-conclusion rules, etc.) was [40].

[2] We borrow the table from [3]. Unlike [3], we use "**1**" for truth, "**0**" for falsity, and "**i**" for the third logical value. As the authors of [3] indicate, the table expresses an idea already present in McCarthy's [34].

Similarly, $\{p\}$ does not entail $p \vee q$, but mc-entails $\{p, q\}$ when disjunction is construed in a way presented in Table 1.2, which expresses the meaning of disjunction in some of Bochvar's logics; see [5].

$\vee$	**0**	**i**	**1**
0	**0**	**i**	**1**
i	**i**	**i**	**i**
1	**1**	**i**	**1**

Table 1.2: *Bochvar's disjunction.*

Mc-entailment, however, exhibits a kind of asymmetry with respect to quantifiers used. As for the condition (3), the clause occurring in the scope of "for each $\mathfrak{M}$" involves universal quantifier in the antecedent and existential quantifier in the consequent. When X mc-entails Y, one expects from X to *consists of* truths, while Y is only required to *contain* a truth. This quantificational heterogeneity shows that X and Y are intuitively understood in different manners. A set of wffs can represent a belief base, but can also represent a *search space*. It seems natural to think of a mc-entailed set as representing a search space. On the other hand, when X mc-entails Y, it seems natural to construe X as a representative of a (potential) belief base.

But what if we are after a relation between sets of wffs each of which represents a search space? At the first step we can consider a relation between sets of wffs, X and Y, fulfilling the following condition:

(4) *for each $\mathfrak{M}$: if at least one wff in X is true in $\mathfrak{M}$, then at least one wff in Y is true in $\mathfrak{M}$.*

Generally speaking, condition (4) expresses the following intuition: if truth can be found in a search space X, then truth can be found in the search space Y.

A natural generalization would be to allow for many search spaces in the antecedent. Let Φ be a family[3] of sets of wffs, and let Y be a set of wffs. We may require Φ and Y be connected according to the following principle:

(5) *for each $\mathfrak{M}$: if, for all $X \in \Phi$, at least one wff in X is true in $\mathfrak{M}$, then at least one wff in Y is true in $\mathfrak{M}$.*

[3] By a family of sets we mean, here and below, a set of sets.

Now the intuition is: if truth can be found in all the search spaces that belong to Φ, then truth can be found in the search space Y as well.

As for condition (5), existential quantifier plays the crucial role both in the antecedent and the consequent. But there is a price: we have jumped to the level of families of sets. Moreover, homogeneity of the domain and the range is lost.

In this chapter I define and investigate a semantic relation between families of sets of wffs and sets of wffs. The basic intuition which underlies the proposed definition is that of the condition (5) above. I dub the relation *generalized entailment*.

1.2 Logical Preliminaries

We consider a formal language for which the concept of well-formed formula (wff) is defined. We use A, B, C, D as metalanguage variables for wffs, and X, Y, W, Z, with subscripts if needed, as metalanguage variables for sets of wffs. The Greek letters Φ, Ψ will refer to families of sets of wffs, that is, sets of sets of wffs. In the metatheory we assume a version of set theory that allows both for sets and classes, and incorporates the Axiom of Choice. We use standard set-theoretic terminology and notation. The expression "iff" abbreviates "if and only if."

As for the general semantic framework, we follow here the idea of [40], yet with some adjustments borrowed from [58].

Definition 1.1 (Partition of the set of wffs)**.** *Let* $\mathtt{Form}$ *be the set of wffs of a formal language. A partition of* $\mathtt{Form}$ *is an ordered pair:*

$$\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$$

where $\mathsf{T_P} \cap \mathsf{U_P} = \emptyset$ *and* $\mathsf{T_P} \cup \mathsf{U_P} = \mathtt{Form}$*.*

We assume that the language considered is supplemented with semantics rich enough to define some concept of truth for wffs. The concept is always relative to some metalogical constructs, such as valuations, models, matrices, etc. The relevant concept of truth determines the *class of admissible partitions* of the language under consideration. The following examples illustrate this.

Example 1.2. Let $\mathtt{Form}_{\mathsf{CPL}}$ be the set of wffs of the language of Classical Propositional Logic (hereafter: CPL). A *Boolean valuation* is a function v that assigns a truth value, $\mathbf{1}$ or $\mathbf{0}$, to each propositional variable and is extended to all wffs in the standard manner by using the Boolean functions corresponding to the connectives.[4]

A partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ of $\mathtt{Form}_{\mathsf{CPL}}$ is *admissible* iff there exists a Boolean valuation v such that:

$$\mathsf{T_P} = \{A \in \mathtt{Form}_{\mathsf{CPL}} : v(A) = \mathbf{1}\}.$$

Thus, for any admissible partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$, the set $\mathsf{T_P}$ comprises all the wffs which are true under the corresponding Boolean valuation v, and (since $\mathsf{U_P} = \mathtt{Form}_{\mathsf{CPL}} \setminus \mathsf{T_P}$), $\mathsf{U_P}$ contains the wffs which are false w.r.t. v.

Example 1.3. We consider the propositional modal logic **S4**. Let $\mathtt{Form}_{\mathbf{S4}}$ be the set of wffs of the language of (propositional) **S4**. The concept of **S4**-Kripke model, as well as the concept of truth of a wff in a world of a model, are defined in the standard manner. We write $(\mathcal{M}, w) \models A$ for "A is true in world w of model $\mathcal{M}$."

A partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ of $\mathtt{Form}_{\mathbf{S4}}$ is *admissible* iff for some **S4**-Kripke model $\mathcal{M} = \langle W, R, V \rangle$ and some $w \in W$:

$$\mathsf{T_P} = \{A \in \mathtt{Form}_{\mathbf{S4}} : (\mathcal{M}, w) \models A\}.$$

Example 1.4. This time we consider First-Order Logic with Identity (hereafter: FOL). The concepts of FOL-model and of truth of a wff in a FOL-model are defined in the standard manner. By $\mathit{Ver}(M)$ we designate the set of all wffs which are true in a FOL-model M.

A partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ of the set of wffs of the language of FOL is *admissible* iff for some FOL-model M:

$$\mathsf{T_P} = \mathit{Ver}(M).$$

Classes of admissible partitions of languages different from these just considered can be defined according to the pattern applied above.[5]

When $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ is an admissible partition, we may think of $\mathsf{T_P}$ as the set of truths of the partition, and of $\mathsf{U_P}$ as the set of untruths of the partition.

---

[4] Alternatively, one can use here the concept of CPL-*valuation*; see Definition 3.1 in Chapter 3.

[5] To be more precise, this is only one of the ways of defining the class of admissible partitions. For some languages admissible partitions can be defined directly (cf. [58] and [29]). It is also possible to "extract" the class of admissible partitions out of a (syntactic) consequence relation determined by a logic (cf. [62]).

Given that the class of admissible partitions is fixed, single-conclusion entailment (or simply: entailment), $\models$, and multiple-conclusion entailment, $\Vdash$, can be defined by:

Definition 1.2 (Single-conclusion entailment; entailment)**.** *$X \models A$ iff there is no admissible partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ such that $X \subseteq \mathsf{T_P}$ and $A \notin \mathsf{T_P}$.*

Definition 1.3 (Multiple-conclusion entailment; mc-entailment)**.** *$X \Vdash Y$ iff there is no admissible partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ such that $X \subseteq \mathsf{T_P}$ and $Y \cap \mathsf{T_P} = \emptyset$.*

For example, in the case of CPL we get: $X \models A$ iff $v(A) = \mathbf{1}$ for every Boolean valuation v such that $v(B) = \mathbf{1}$ for any $B \in X$. As for mc-entailment, we have: $X \Vdash Y$ iff for each Boolean valuation v in which $v(B) = \mathbf{1}$ for every $B \in X$, there exists $A \in Y$ such that $v(A) = \mathbf{1}$.

In what follows, we assume that the language for which we define generalized entailment and the remaining concepts, is an arbitrary but fixed formal language satisfying the general conditions specified in this section. By admissible partitions we mean admissible partitions of the set of wffs of the language.

1.3 Definition of Generalized Entailment

Generalized entailment (g-entailment for short) is a relation between a family of sets of wffs on the one hand, and a set of wffs on the other. We use $|\!|\!\models$ as the symbol for g-entailment.

Definition 1.4 (Generalized entailment; g-entailment)**.** *$\Phi \,|\!|\!\models Y$ iff for each admissible partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ such that:*

$(\star)$ *for each $X \in \Phi : X \cap \mathsf{T_P} \neq \emptyset$*

it holds that $Y \cap \mathsf{T_P} \neq \emptyset$.

The proposed definition of g-entailment expresses, in the current conceptual setting, the idea that lies behind condition (5) specified in section 1.1 above.

1.3.1 Some Examples

Some examples can be helpful.

Example 1.5. As for CPL, we have: $\Phi \,|\!|\!\models Y$ iff there is no Boolean valuation v such that:

- for each $X \in \Phi$, $v(A) = \mathbf{1}$ for some $A \in X$, and $v(B) = \mathbf{0}$ for all $B \in Y$.

For instance, the following holds (p, q, r, s, t, u are, here and below, propositional variables):

$$\{\{p \vee q \to s \vee r, p \vee q \to s \vee t\}, \{p, q\}\} \,|\!|\!\models \{s, r, t\}$$

Example 1.6. Consider the case of FOL. We have: $\Phi \,|\!|\!\models Y$ iff for each FOL-model M:

- if $X \cap \mathit{Ver}(M) \neq \emptyset$ for each $X \in \Phi$, then $Y \cap \mathit{Ver}(M) \neq \emptyset$.

For instance ($\mathtt{P}, \mathtt{S}$ are one-place predicates, and $\mathtt{a}, \mathtt{b}$ are individual constants):

$$\{\{\exists x \mathtt{P}x, \exists x \mathtt{S}x\}, \{\forall x(\mathtt{P}x \vee \mathtt{S}x \to x = \mathtt{a} \vee x = \mathtt{b}\}\} \,|\!|\!\models \{\mathtt{Pa}, \mathtt{Pb}, \mathtt{Sa}, \mathtt{Sb}\}$$

Note that $\{\forall x(\mathtt{P}x \vee \mathtt{S}x \to x = \mathtt{a} \vee x = \mathtt{b}\}$ is a singleton set. However, it is not excluded that Φ contains singleton sets.

1.4 Basic Properties of Generalized Entailment

Recall that $|\!|\!\models$ is a relation between a family of sets of wffs and a set of wffs. Interestingly enough, $|\!|\!\models$ still behaves in a "consequence-like" manner.

Proposition 1.1. *If $X \in \Phi$, then $\Phi \,|\!|\!\models X$.*

Proof. Suppose otherwise. Then there exists an admissible partition, P, such that both $X \cap \mathsf{T}_\mathsf{P} = \emptyset$ and $X \cap \mathsf{T}_\mathsf{P} \neq \emptyset$. A contradiction. □

Proposition 1.2. *If $\Phi \,|\!|\!\models Y$ and $\Phi \subseteq \Psi$, then $\Psi \,|\!|\!\models Y$.*

Proof. Suppose that $\Psi \,|\!|\!\not\models Y$. Thus there exists an admissible partition, P, such that $Y \cap \mathsf{T}_\mathsf{P} = \emptyset$ and for each $Z \in \Psi : Z \cap \mathsf{T}_\mathsf{P} \neq \emptyset$. As $\Phi \subseteq \Psi$, it follows that $\Phi \,|\!|\!\not\models Y$. □

Proposition 1.3.
If $\Phi \,|\!|\!\models Y$ and $\Psi \,|\!|\!\models X$ for every $X \in \Phi$, then $\Psi \,|\!|\!\models Y$.

Proof. Suppose that $\Psi \,|\!|\!\not\models Y$. So for some admissible partition, P, we have $Y \cap \mathsf{T}_\mathsf{P} = \emptyset$ and $Z \cap \mathsf{T}_\mathsf{P} \neq \emptyset$ for any $Z \in \Psi$. Since $\Psi \,|\!|\!\models X$ for every $X \in \Phi$, it follows that $X \cap \mathsf{T}_\mathsf{P} \neq \emptyset$ for each $X \in \Phi$. Hence $Y \cap \mathsf{T}_\mathsf{P} \neq \emptyset$, as $\Phi \,|\!|\!\models Y$. A contradiction. □

1.5 Special Cases and Cut

Let us introduce:

Definition 1.5 (Safeset). *Y is a safeset iff $Y \cap \mathsf{T}_\mathsf{P} \neq \emptyset$ for each admissible partition $\mathsf{P} = \langle \mathsf{T}_\mathsf{P}, \mathsf{U}_\mathsf{P} \rangle$.*

A set containing a valid[6] wff is a safeset. But there exist safesets which do not contain valid wffs. For instance, $\{p, \neg p\}$ is a safeset in view of CPL.

Clearly, we have:

Corollary 1.1. *A safeset is g-entailed by any family of sets of wffs.*

Proposition 1.4. *If $\emptyset \mid\mid\models Y$, then Y is a safeset.*

Proof. Assume that $\emptyset \mid\mid\models Y$. Let P be an arbitrary but fixed admissible partition. The condition ($\star$) of Definition 1.4 is (trivially) true w.r.t. $\Phi = \emptyset$. We have:

($\star'$) for each $X \in \emptyset$: $X \cap \mathsf{T}_\mathsf{P} \neq \emptyset$.

Hence $Y \cap \mathsf{T}_\mathsf{P} \neq \emptyset$. But P is an arbitrary admissible partition. □

Thus the empty set g-entails only safesets. But if the empty set is an element of a family of sets of wffs, the family g-entails any set of wffs. This is due to:

Proposition 1.5. *If $\emptyset \in \Phi$, then $\Phi \mid\mid\models Y$ for any set of wffs Y.*

Proof. Assume that $\Phi \mid\mid\not\models Y$ for some set of wffs Y. Thus there exists an admissible partition, P, for which the following condition holds:

$$\textit{for each } X \in \Phi : X \cap \mathsf{T}_\mathsf{P} \neq \emptyset \tag{1.1}$$

However, $\emptyset \in \Phi$ and hence the condition (1.1) yields:

$$\emptyset \cap \mathsf{T}_\mathsf{P} \neq \emptyset \tag{1.2}$$

which is impossible. □

G-entailment has a property akin to cut:

[6] A wff A is *valid* iff $A \in \mathsf{T}_\mathsf{P}$ for each admissible partition $\mathsf{P} = \langle \mathsf{T}_\mathsf{P}, \mathsf{U}_\mathsf{P} \rangle$ of the set of wffs of the language.

Proposition 1.6. *If* $\Psi \mid\mid\models X$ *and* $\Phi \cup \{X\} \mid\mid\models Y$*, then* $\Psi \cup \Phi \mid\mid\models Y$*.*

Proof. Assume that $\Psi \mid\mid\models X$ and $\Phi \cup \{X\} \mid\mid\models Y$, but $\Psi \cup \Phi \mid\mid\not\models Y$. If follows that there exists an admissible partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ such that $Z \cap \mathsf{T_P} \neq \emptyset$ for each $Z \in \Psi \cup \Phi$, and $Y \cap \mathsf{T_P} = \emptyset$. As $\Psi \mid\mid\models X$, we have $X \cap \mathsf{T_P} \neq \emptyset$. But $\Phi \cup \{X\} \mid\mid\models Y$ and hence $Y \cap \mathsf{T_P} \neq \emptyset$. A contradiction. □

As an immediate consequence of Proposition 1.6 we get:

Corollary 1.2. *If* $\Phi \mid\mid\models X$ *and* $\Phi \cup \{X\} \mid\mid\models Y$*, then* $\Phi \mid\mid\models Y$*.*

1.6 Generalized Entailment versus Entailment and Multiple-Conclusion Entailment

Both entailment and mc-entailment are definable in terms of g-entailment. However, we need an auxiliary concept.

Definition 1.6 (Disperse). $\breve{X} =_{df} \{\{A\} : A \in X\}$.

$\breve{X}$ is thus the family of singleton sets based on the elements of X. The family $\breve{X}$ may be called the *disperse* of set X. Observe that $\breve{\emptyset} = \emptyset$.

The following holds:

Corollary 1.3. *For every admissible partition* $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$*:* $X \subseteq \mathsf{T_P}$ *iff for each* $Z \in \breve{X} : Z \subseteq \mathsf{T_P}$*.*

Proof. Just notice that $X \subseteq \mathsf{T_P}$ iff $B \in \mathsf{T_P}$ for any $B \in X$. □

By Corollary 1.3 we obtain that g-entailment and entailment are linked in the way described by:

Proposition 1.7. $X \models A$ *iff* $\breve{X} \mid\mid\models \{A\}$*.*

As for mc-entailment, again by Corollary 1.3, we have:

Proposition 1.8. $X \mid\models Y$ *iff* $\breve{X} \mid\mid\models Y$*.*

1.7 Generalized Entailment and Choices

Let us come back to Example 1.5 presented in section 1.3.1. As we remarked, the following holds:

$$\{\{p \vee q \rightarrow s \vee r, p \vee q \rightarrow s \vee t\}, \{p, q\}\} \mathrel{\|\!\models} \{s, r, t\}$$

For brevity, we designate $\{p \vee q \rightarrow s \vee r, p \vee q \rightarrow s \vee t\}$ by X_1, $\{p, q\}$ by X_2, and $\{s, r, t\}$ by Y. Thus $\{X_1, X_2\} \mathrel{\|\!\models} Y$. Now let us consider the following sets of wffs:

$$Z_1 = \{p \vee q \rightarrow s \vee r, p\}$$

$$Z_2 = \{p \vee q \rightarrow s \vee r, q\}$$

$$Z_3 = \{p \vee q \rightarrow s \vee t, p\}$$

$$Z_4 = \{p \vee q \rightarrow s \vee t, q\}$$

Each Z_i, where $1 \leq i \leq 4$, is a set that contains exactly one representative of X_1 and exactly one representative of X_2. Observe that we have:

$$Z_i \mathrel{\|\!\models} Y$$

that is, each Z_i $(1 \leq i \leq 4)$ mc-entails Y. In other words, any set which contains exactly one representative of X_1 and exactly one representative of X_2 mc-entails Y.

The above observation can be generalized and then turned into an equivalence, but some caution is needed. We have to express in exact terms the idea of a set which contains *exactly one representative* of each non-empty set belonging to a previously given family of sets. This can be done in many ways. In the next section we present a solution which, additionally, will be used in defining the concept of constructive generalized entailment in Chapter 2.

1.7.1 $\mathtt{ch}^{\otimes}(\Phi)$-sets and $\mathtt{ch}(\Phi)$-sets

We introduce, first, the following technical concept[7] ($\times$ stands here for the sign of Cartesian product):

[7] I am indebted to Jerzy Pogonowski for his suggestion to use the concept for the purposes of analysis of g-entailment.

Definition 1.7.

$$X^{\otimes} = \begin{cases} X \times \{X\} & \textit{if } X \neq \emptyset, \\ \emptyset & \textit{if } X = \emptyset. \end{cases}$$

Clearly, we have:

Corollary 1.4. *If $X \neq Z$, then $X^{\otimes} \cap Z^{\otimes} = \emptyset$.*

Definition 1.8. $\Phi^{\otimes} =_{df} \{X^{\otimes} : X \in \Phi\}$.

Obviously, if $\Phi = \emptyset$, then $\Phi^{\otimes} = \emptyset$. (To see this it suffices to observe that $\{X^{\otimes} : X \in \emptyset\} = \emptyset$.) The following holds:

Corollary 1.5. *If $\Phi^{\otimes} \neq \emptyset$ and $\emptyset \notin \Phi^{\otimes}$, then there exists a set γ such that γ comprises exactly one element $\langle A, X\rangle$ of each $X^{\otimes} \in \Phi^{\otimes}$.*

Proof. By the Axiom of Choice (observe that Corollary 1.4 warrants that the elements of $\Phi^{\otimes}$ are pairwise disjoint). □

Our second technical concept is given by:

Definition 1.9 ($\mathtt{ch}^{\otimes}(\Phi)$-set)**.** *$\gamma$ is a $\mathtt{ch}^{\otimes}(\Phi)$-set iff*

1. *$\gamma \subseteq \bigcup \Phi^{\otimes}$ and*
2. *for each $X^{\otimes} \in \Phi^{\otimes}$ such that $X^{\otimes} \neq \emptyset$ there exists exactly one $\langle A, X\rangle \in X^{\otimes}$ such that $\langle A, X\rangle \in \gamma$.*

The following corollaries will be useful:

Corollary 1.6. *If $\Phi = \emptyset$, then $\emptyset$ is the only $\mathtt{ch}^{\otimes}(\Phi)$-set.*

Proof. If $\Phi = \emptyset$, then $\Phi^{\otimes} = \emptyset$. Hence clause (*2*) of Definition 1.9 is fulfilled by $\emptyset$ (since there is no $X^{\otimes} \in \Phi^{\otimes}$ such that $X^{\otimes} \neq \emptyset$). Clearly, $\bigcup \emptyset = \emptyset$, and $\emptyset$ is the only subset of $\bigcup \Phi^{\otimes}$. □

Corollary 1.7. *If $\Phi = \{\emptyset\}$, then $\emptyset$ is the only $\mathtt{ch}^{\otimes}(\Phi)$-set.*

Proof. $\emptyset$ is the only element of Φ. Hence there is no $X^{\otimes} \in \Phi^{\otimes}$ such that $X^{\otimes} \neq \emptyset$. □

Corollary 1.8. *If $\Phi \neq \emptyset$ and $\Phi \neq \{\emptyset\}$, then $\Phi^{\otimes} \neq \emptyset$.*

Proof. If $\Phi \neq \emptyset$ and $\Phi \neq \{\emptyset\}$, then there exists $X \in \Phi$ such that $X \neq \emptyset$. Clearly, $X^{\otimes} \neq \emptyset$. On the other hand, $X^{\otimes} \in \Phi^{\otimes}$. □

One can prove:

Proposition 1.9. *For each family of sets* Φ *there exists at least one* $\mathtt{ch}^{\otimes}(\Phi)$*-set.*

Proof. If $\Phi = \emptyset$ or $\Phi = \{\emptyset\}$, then, by corollaries 1.6 and 1.7, $\emptyset$ is the only $\mathtt{ch}^{\otimes}(\Phi)$-set. If $\Phi \neq \emptyset$ and $\Phi \neq \{\emptyset\}$, then $\Phi^{\otimes} \neq \emptyset$ by Corollary 1.8, at least one element of $\Phi^{\otimes}$ is a non-empty set, and all the elements of $\Phi^{\otimes}$ are disjoint. Assume that $\emptyset \notin \Phi^{\otimes}$. The existence of $\mathtt{ch}^{\otimes}(\Phi)$-set is now warranted by Corollary 1.5. Assume that $\emptyset \in \Phi^{\otimes}$. Thus $\emptyset \in \Phi$. Since, by assumption, $\Phi \neq \{\emptyset\}$, we move to $\Psi = \Phi \setminus \{\emptyset\}$. Clearly, the sets in $\Psi^{\otimes}$ are non-empty and disjoint. Thus, by Corollary 1.5 again, there exists a $\mathtt{ch}^{\otimes}(\Psi)$-set, say, γ. But, obviously, γ is also a $\mathtt{ch}^{\otimes}(\Phi)$-set. □

A $\mathtt{ch}^{\otimes}(\Phi)$-set is a set of ordered pairs. We take into account the first projection of the set.

Definition 1.10 (First projection)**.** *Let* γ *be a* $\mathtt{ch}^{\otimes}(\Phi)$*-set.*

$$\gamma^1 =_{df} \{A : \langle A, X\rangle \in \gamma\}.$$

Now we define the basic technical concept.

Definition 1.11 ($\mathtt{ch}$-set)**.** *Z is a* $\mathtt{ch}(\Phi)$*-set iff* $Z = \gamma^1$ *for some* $\mathtt{ch}^{\otimes}(\Phi)$*-set* γ*.*

A $\mathtt{ch}(\Phi)$-set is a set comprising exactly one *representative* of each non-empty set belonging to Φ. One should not confuse the existence of exactly one representative of each set belonging to a family of sets with the existence of a system of distinct representatives of the family.[8] The representatives of distinct sets in a $\mathtt{ch}$-set need not be distinct.

Example 1.7. Let $\Phi = \{X_1, X_2\}$, where $X_1 = \{p, q\}$ and $X_2 = \{p, r\}$. The following are $\mathtt{ch}^{\otimes}$-sets:

$$\{\langle p, X_1\rangle, \langle p, X_2\rangle\},$$

$$\{\langle p, X_1\rangle, \langle r, X_2\rangle\},$$

$$\{\langle q, X_1\rangle, \langle p, X_2\rangle\},$$

$$\{\langle q, X_1\rangle, \langle r, X_2\rangle\}.$$

[8] As it is well-known, a system of distinct representatives – a transversal of a family of sets – does not always exist; cf., e.g., [55], Chapter 8.

Thus the family of $\mathtt{ch}(\Phi)$-sets comprises: $\{p\}$, $\{p, r\}$, $\{p, q\}$, $\{q, r\}$. As for the $\mathtt{ch}(\Phi)$-set $\{p\}$, p is the representative of X_1 and is the representative of X_2.

In the light of Proposition 1.9, the following holds:

Proposition 1.10. *For each family of sets Φ there exists at least one $\mathtt{ch}(\Phi)$-set.*

Let us also note:

Proposition 1.11. *Let $A \in X$ for some $X \in \Phi$. There exists at least one $\mathtt{ch}(\Phi)$-set such that A belongs to this set.*

Proof. The family Φ can be displayed as the union of the following families of sets:

$$\Phi_1 = \{X \in \Phi : A \in X\}$$

$$\Phi_2 = \{X \in \Phi : A \notin X\}$$

where $\Phi_1 \cap \Phi_2 = \emptyset$. By Proposition 1.9, there exists a $\mathtt{ch}^{\otimes}(\Phi_2)$-set, say, γ. Let us define a set δ by the condition:

$$\langle B, Y\rangle \in \delta \ \textit{ iff } B = A \textit{ and } Y \in \Phi_1$$

Let $\zeta = \gamma \cup \delta$. As $\Phi_1 \cap \Phi_2 = \emptyset$, we also have $\gamma \cap \delta = \emptyset$. It is easily seen that $\gamma \cup \delta$ is a $\mathtt{ch}^{\otimes}(\Phi)$-set. Thus $(\gamma \cup \delta)^1$ is a $\mathtt{ch}(\Phi)$-set. Obviously, $A \in (\gamma \cup \delta)^1$. □

1.7.2 Generalized Entailment and ch(Φ)-sets

A $\mathtt{ch}(\Phi)$-set can be intuitively understood as a "choice set": we choose from each non-empty set that belongs to Φ its representative. Thus quantifying over all $\mathtt{ch}(\Phi)$-sets amounts to quantifying over all possible choices of this kind. In this section we show that, for any family of non-empty sets of wffs, being g-entailed by the family amounts to being mc-entailed by each "choice set" associated with the family, that is, by any ch-set of the family.

Theorem 1.1.
Let $\emptyset \notin \Phi$. Then $\Phi \Vert\!\models Y$ iff $Z \Vert\!\models Y$ for each $\mathtt{ch}(\Phi)$-set Z.

Proof. ($\Rightarrow$) Assume that $\Phi \Vvdash Y$. Suppose that that $Z \Vdash\!\!\!\!/\; Y$ for some $\mathtt{ch}(\Phi)$-set Z. Thus there exists an admissible partition, P, such that $Z \subseteq \mathsf{T_P}$ and $Y \cap \mathsf{T_P} = \emptyset$.

Assume that $Z = \emptyset$. Hence $\Phi = \emptyset$ or $\Phi = \{\emptyset\}$. However, by assumption $\emptyset \notin \Phi$. Thus $\Phi = \emptyset$. As $\Phi \Vvdash Y$, by Proposition 1.4 we get that Y is a safeset. But $Y \cap \mathsf{T_P} = \emptyset$. A contradiction.

Now assume that $Z \neq \emptyset$. Since Z is a $\mathtt{ch}(\Phi)$-set, it contains elements of each set in Φ (recall that, by assumption, these sets are non-empty). Hence for each $X \in \Phi$ we have $X \cap \mathsf{T_P} \neq \emptyset$. But $Y \cap \mathsf{T_P} = \emptyset$. It follows that $\Phi \not\Vvdash Y$. A contradiction.
($\Leftarrow$) Assume that $\Phi \neq \emptyset$. Suppose that $\Phi \not\Vvdash Y$. By assumption, $\emptyset \notin \Phi$. Thus for some admissible partition, P, we have $X \cap \mathsf{T_P} \neq \emptyset$ for each $X \in \Phi$, and $Y \cap \mathsf{T_P} = \emptyset$. Recall that, again by assumption, Φ comprises non-empty sets. We assign to each set $X \in \Phi$ the corresponding set X^* by:

$$X^* = X \cap \mathsf{T_P}$$

Let Φ^* be the family of all X^*-sets defined in the above manner. By Proposition 1.9, Φ^* has a $\mathtt{ch}^{\otimes}(\Phi^*)$-set, say, δ. Observe that $\delta \neq \emptyset$ and $\delta^1 \subseteq \mathsf{T_P}$. We define a set γ by:

$$\gamma = \{\langle A, X\rangle \in \Phi^{\otimes} : \langle A, X^*\rangle \in \delta\}$$

It is clear that γ is a $\mathtt{ch}^{\otimes}(\Phi)$-set. Moreover, $\gamma^1 = \delta^1$. So there exists a $\mathtt{ch}(\Phi)$-set, namely γ^1, such that $\gamma^1 \subseteq \mathsf{T_P}$. Hence $Z \not\Vdash Y$ for some $\mathtt{ch}(\Phi)$-set Z.

Finally, assume that $\Phi = \emptyset$. It follows that $\emptyset$ is the only $\mathtt{ch}(\Phi)$-set. Suppose that $\Phi \not\Vvdash Y$. Thus there exists an admissible partition, P, such that $Y \cap \mathsf{T_P} = \emptyset$. Therefore $\emptyset \not\Vdash Y$. □

Remark that the assumption "$\emptyset \notin \Phi$" is a necessary one. As we have shown (cf. Proposition 1.5), a family of sets that includes the empty set g-entails any set of wffs.

Chapter 2

Constructive Generalized Entailment

2.1 Constructive Generalized Entailment: Intuitions

In this chapter I define and analyse a semantic relation between families of sets of wffs and sets of wffs, dubbed *constructive generalized entailment*. The intuitions which lie behind the proposed definition are slightly different from those which underlie the concept of generalized entailment. However, constructive generalized entailment is a special case of generalized entailment.

The terminology and notation used in this chapter are these of Chapter 1.

In order to explain the intuitions which lie behind the concept of constructive generalized entailment, let us come back to Example 1.5 presented in section 1.3.1 of Chapter 1.

Let $\Phi = \{X_1, X_2\}$, where $X_1 = \{p \vee q \to s \vee r, p \vee q \to s \vee t\}$ and $X_2 = \{p, q\}$. We have $\Phi \;\|\!\models \{s, r, t\}$. The respective $\mathtt{ch}(\Phi)$-sets are:

$$\{p \vee q \to s \vee r, p\}$$

$$\{p \vee q \to s \vee r, q\}$$

$$\{p \vee q \to s \vee t, p\}$$

$$\{p \vee q \to s \vee t, q\}$$

Each of the above $\mathtt{ch}(\Phi)$-sets mc-entails the set $\{s, r, t\}$. But observe that none of the above $\mathtt{ch}(\Phi)$-sets entails a single wff in $\{s, r, t\}$!

However, we also have:

$$\Phi \ \Vert\!\models \ \{s \vee r, s \vee t\} \tag{2.1}$$

In the case of (2.1) g-entailment is *constructive*: for each $\mathtt{ch}(\Phi)$-set there exists a single wff in $\{s \vee r, s \vee t\}$ which is entailed by the $\mathtt{ch}(\Phi)$-set. Here is another example of this kind. Let $\Psi = \Phi \cup \{\neg s\}$. We have $\Psi \ \Vert\!\models \ \{r, t\}$. Each $\mathtt{ch}(\Psi)$-set results from a $\mathtt{ch}(\Phi)$-set by adding $\neg s$. It is easily visible that any $\mathtt{ch}(\Psi)$-set either entails r or entails t.

2.2 Definition of Constructive Generalized Entailment

Constructive generalized entailment, or cg-entailment for short, is a relation between a family of sets of wffs and a set of wffs. We use $\mid\!\rhd$ as the symbol for cg-entailment.

Definition 2.1 (Constructive generalized entailment; cg-entailment). $\Phi \mid\!\rhd Y$ *iff for each* $\mathtt{ch}(\Phi)$*-set* X *there exists* $A \in Y$ *such that* $X \models A$.

Thus $\Phi \mid\!\rhd Y$ holds just in case each $\mathtt{ch}(\Phi)$-set entails some wff in Y. Remark that different $\mathtt{ch}(\Phi)$-sets may entail different elements of Y.

By Definition 1.2 we get:

Corollary 2.1. $\Phi \mid\!\rhd Y$ *iff for each* $\mathtt{ch}(\Phi)$*-set* X *there exists* $A \in Y$ *such that for any admissible partition* $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ *the following condition holds:*

(∗) *if* $X \subseteq \mathsf{T_P}$*, then* $A \in \mathsf{T_P}$*.*

2.2.1 Examples of Constructive Generalized Entailment

As it has been explained in section 2.1, the following hold:

$$\{\{p \vee q \to s \vee r, p \vee q \to s \vee t\}, \{p, q\}\} \mid\!\rhd \{s \vee r, s \vee t\} \tag{2.2}$$

$$\{\{p \vee q \to s \vee r, p \vee q \to s \vee t\}, \{p, q\}, \{\neg s\}\} \mid\!\rhd \{r, t\} \tag{2.3}$$

Here are further CPL-examples. Let:

$$Y = \{s \to p, \neg s \to q \vee r, q \leftrightarrow u\}$$

We have:

$$\breve{Y} \cup \{\{s, \neg s\}, \{u, \neg u\}\} \mathrel{|\!\!\!>} \{p, q, r\} \tag{2.4}$$

Let

$$Z = \{p \vee q \vee r, s \rightarrow p, \neg s \rightarrow \neg r, q \leftrightarrow t \vee u\}$$

The following holds:

$$\breve{Z} \cup \{\{s, \neg s\}, \{t, \neg t\}, \{u, \neg u\}\} \mathrel{|\!\!\!>} \{p, q, r\} \tag{2.5}$$

Let us now switch to FOL. We have:

$$\{\{\forall x(\mathtt{P}x \leftrightarrow \mathtt{S}x \vee (\mathtt{T}x \wedge \mathtt{U}x))\}, \{\mathtt{Sa}, \neg\mathtt{Sa}\}, \{\mathtt{Ta}, \neg\mathtt{Ta}\}, \{\mathtt{Ua}, \neg\mathtt{Ua}\}\} \mathrel{|\!\!\!>} \{\mathtt{Pa}, \neg\mathtt{Pa}\} \tag{2.6}$$

2.3 Basic Properties of Constructive Generalized Entailment

Clearly, the following is true:

Corollary 2.2. *If* $\Phi \mathrel{|\!\!\!>} Y$*, then* $Y \neq \emptyset$*.*

Observe that being a safeset is not sufficient for being cg-entailed by a family of wffs. For instance, $\{p, \neg p\}$ is a safeset (w.r.t. CPL), but $\{p, \neg p\}$ is not cg-entailed by the singleton family $\{\{p \vee \neg p\}\}$ (again, in CPL). The situation is different in the case of g-entailment (cf. Corollary 1.1).

Similarly as g-entailment, also cg-entailment behaves in a "consequence-like" manner.

Proposition 2.1. $\Phi \mathrel{|\!\!\!>} Y$ *for each* $Y \in \Phi$ *such that* $Y \neq \emptyset$*.*

Proof. It suffices to observe that if $Y \in \Phi$ and $Y \neq \emptyset$, then each $\mathtt{ch}(\Phi)$-set contains an element of Y. □

Proposition 2.2. *If* $\Phi \mathrel{|\!\!\!>} Y$ *and* $\Phi \subseteq \Psi$*, then* $\Psi \mathrel{|\!\!\!>} Y$*.*

Proof. It suffices to observe that if $\Phi \subseteq \Psi$, then each $\mathtt{ch}(\Psi)$-set has a subset being a $\mathtt{ch}(\Phi)$-set. □

Proposition 2.3. *If* $\Phi \mathrel{|\!\!\!>} Y$ *and* $\Psi \mathrel{|\!\!\!>} X$ *for each* $X \in \Phi$*, then* $\Psi \mathrel{|\!\!\!>} Y$*.*

Proof. Let W be an arbitrary but fixed $\mathtt{ch}(\Psi)$-set. By assumption, $\Psi \mathrel{|\!\!\rhd} X$ for each $X \in \Phi$. Thus for any $X \in \Phi$, the set $X_{(W)}$ defined by:

$$X_{(W)} = \{B \in X : W \models B\}$$

is non-empty. Let $\Phi_{(W)}$ be the family of all $X_{(W)}$-sets defined in the above manner. By Proposition 1.9, the family $\Phi_{(W)}$ has a $\mathtt{ch}^{\otimes}(\Phi_{(W)})$-set, say, μ. Thus μ^1 is a $\mathtt{ch}(\Phi_{(W)})$-set. Moreover, we have $W \models D$ for each $D \in \mu^1$.

We define:

$$\varrho = \{\langle C, X\rangle \in \Phi^{\otimes} : \langle C, X_{(W)}\rangle \in \mu\}$$

Clearly, ϱ is a $\mathtt{ch}^{\otimes}(\Phi)$-set and thus ϱ^1 is a $\mathtt{ch}(\Phi)$-set. Observe that $\varrho^1 = \mu^1$. As $\Phi \mathrel{|\!\!\rhd} Y$, there exists $A \in Y$ such that $\varrho^1 \models A$. But, since $\varrho^1 = \mu^1$, we have $W \models D$ for each $D \in \varrho^1$. Therefore $W \models A$. Hence $\Psi \mathrel{|\!\!\rhd} Y$. □

Note, however, that if $\emptyset \in \Phi$, then, by Corollary 2.2, it is not the case that $\Phi \mathrel{|\!\!\rhd} \emptyset$.

2.4 Constructive Generalized Entailment and Cut

It should be noted that cg-entailment, similarly as g-entailment, has a feature analogous to cut.

Proposition 2.4. *If* $\Psi \mathrel{|\!\!\rhd} X$ *and* $\Phi \cup \{X\} \mathrel{|\!\!\rhd} Y$*, then* $\Psi \cup \Phi \mathrel{|\!\!\rhd} Y$*.*

Proof. Assume that $X \in \Phi$. Since $\Phi \cup \{X\} \mathrel{|\!\!\rhd} Y$ holds, we get $\Psi \cup \Phi \mathrel{|\!\!\rhd} Y$ by Proposition 2.2.

Assume that $X \notin \Phi$. Let Z be an arbitrary but fixed $\mathtt{ch}(\Psi \cup \Phi)$-set. Thus $Z = \gamma^1$ for some $\mathtt{ch}^{\otimes}(\Psi \cup \Phi)$-set γ. We define the following sets:

$$\gamma_{\Psi} = \{\langle C, W\rangle \in \gamma : W \in \Psi\}$$

$$\gamma_{\Phi} = \{\langle C, W\rangle \in \gamma : W \in \Phi\}$$

γ_{Ψ} is a $\mathtt{ch}^{\otimes}(\Psi)$-set, and γ_{Φ} is a $\mathtt{ch}^{\otimes}(\Phi)$-set. Thus $(\gamma_{\Psi})^1$ is a $\mathtt{ch}(\Psi)$-set, and $(\gamma_{\Phi})^1$ is a $\mathtt{ch}(\Phi)$-set.

Since, by assumption, $\Psi \mathrel{|\!\!\rhd} X$, there exists an element of X, say, A, such that $(\gamma_{\Psi})^1 \models A$. Let us define:

$$\gamma^*_{\Phi} = \gamma_{\Phi} \cup \{\langle A, X\rangle\}$$

γ_Φ^* is a $\mathtt{ch}^\otimes(\Phi \cup \{X\})$-set (since $X \notin \Phi$) and thus $(\gamma_\Phi^*)^1$ is a $\mathtt{ch}(\Phi \cup \{X\})$-set. Clearly, $A \in (\gamma_\Phi^*)^1$. By assumption, $\Phi \cup \{X\} \mathrel{|\!\!\rhd} Y$, and hence $(\gamma_\Phi^*)^1 \models B$ for some $B \in Y$. On the other hand, we have:

$$(\gamma_\Phi^*)^1 = (\gamma_\Phi)^1 \cup \{A\}$$

and $(\gamma_\Psi)^1 \models A$. Hence $(\gamma_\Psi)^1 \cup (\gamma_\Phi)^1 \models B$. Since $((\gamma_\Psi)^1 \cup (\gamma_\Phi)^1) \subseteq Z$, it follows that $Z \models B$. Therefore $\Psi \cup \Phi \mathrel{|\!\!\rhd} Y$. □

As an immediate consequence of Proposition 2.4 we get:

Corollary 2.3. *$\Phi \mathrel{|\!\!\rhd} X$ and $\Phi \cup \{X\} \mathrel{|\!\!\rhd} Y$, then $\Phi \mathrel{|\!\!\rhd} Y$.*

2.5 Constructive Generalized Entailment versus Entailment and Generalized Entailment

One can prove that entailment of a wff A from a set of wffs X amounts to cg-entailment of the singleton set $\{A\}$ from the disperse of X.

Proposition 2.5. *$X \models A$ iff $\breve{X} \mathrel{|\!\!\rhd} \{A\}$.*

Proof. By Corollary 1.3. Observe that X is the only $\mathtt{ch}(\breve{X})$-set. □

Let us now prove that cg-entailment is a special case of g-entailment.

Proposition 2.6. *If $\Phi \mathrel{|\!\!\rhd} Y$, then $\Phi \mathrel{||\!\!\models} Y$.*

Proof. Assume that $\Phi \mathrel{|\!\!\rhd} Y$.

Let $\emptyset \notin \Phi$. Suppose that $\Phi \mathrel{||\!\!\not\models} Y$. By Theorem 1.1, there exists a $\mathtt{ch}(\Phi)$-set, say, Z, such that $Z \mathrel{|\!\!\not\models} Y$. It follows that there is no $A \in Y$ such that $Z \models A$ and hence it is not the case that $\Phi \mathrel{|\!\!\rhd} Y$. A contradiction.

Let $\emptyset \in \Phi$. Thus, by Proposition 1.5, $\Phi \mathrel{||\!\!\models} Y$. □

Note that the converse of Proposition 2.6 does not hold. The example presented at the beginning of this Chapter illustrates this. Here is another. We have:

$$\{\{p \vee q\}\} \mathrel{||\!\!\models} \{p, q\}$$

but we do not have:

$$\{\{p \vee q\}\} \mathrel{|\!\!\rhd} \{p, q\}.$$

A more sophisticated counterexample is: we have $\{\emptyset\} \,|\!|\!\models \{p, \neg p\}$ (as $\{\emptyset\}$ g-entails every set), but we do not have $\{\emptyset\} \mathrel{|\!\!\vartriangleright} \{p, \neg p\}$ (since $\emptyset$ is the only $\mathtt{ch}(\{\emptyset\})$-set, and neither p nor $\neg p$ is entailed by $\emptyset$).

However, cg-entailment and g-entailment coincide on singleton sets provided that $\emptyset \notin \Phi$.

Corollary 2.4. *Let $\emptyset \notin \Phi$. Then $\Phi \,|\!|\!\models \{A\}$ iff $\Phi \mathrel{|\!\!\vartriangleright} \{A\}$.*

Proof. By Theorem 1.1 and the fact that mc-entailment of $\{A\}$ and entailment of A coincide. □

2.6 Final Remarks

Constructive generalized entailment is a relation between a family of sets of wffs and a set of wffs. When one interprets the respective sets of wffs as sets of principal possible answers to questions, some concept of interrogative entailment can be explicated in terms of cg-entailment (see [61]).[9] It can also be shown that, at the propositional level, cg-entailment simulates inquisitive entailment; see, again, [61]. However, the logic of questions as well as Inquisitive Semantics are beyond the scope of this book.

No full-fledged proof theory for cg-entailment has been elaborated so far. Yet, it can be shown (cf. [61]) that the so-called erotetic search scenarios carry information about concrete cases of cg-entailment.[10] A dedicated software which enables, *inter alia*, automatic generation of erotetic search scenarios out of a predefined set of such scenarios, has been developed (cf. [9]). This brings a computational perspective into research on cg-entailment.

[9] According to the idea which is present in the majority of logical theories of questions (for surveys, see, e.g., [20] and [60]), a question "offers" a set of "alternatives," and the alternatives are expressed by principal possible answers to the question, usually labelled *direct answers* to it. Principal possible answers/direct answers, in turn, are these possible answers to a question that are "optimal" in the sense that they provide neither more nor less information than it is requested. Being true *is not* a prerequisite of being a direct answer. For details, see, e.g., [60].

[10] For erotetic search scenarios see, e.g., [56] or [58], Part III.

Chapter 3

A Uniform Account of Proofs and Refutations. The Propositional Case

3.1 Introduction

Consider a formal language supplemented with a bivalent semantics rich enough to define some concept of truth of a well-formed formula (henceforth: wff) in a model. The expression "model" is used here as a cover term; depending on the particular form of the language, models are valuations of some kind, relational structures, and so on. Usually, a formal language has many models of a given kind. When a non-empty class of models, $\mathtt{CM}$, is fixed, the set of all wffs of the language splits, first, into two disjoint subsets: $\mathtt{Val}^{\mathtt{CM}}$ and $\mathtt{NVal}^{\mathtt{CM}}$. The set $\mathtt{Val}^{\mathtt{CM}}$ comprises all the wffs which are *valid* w.r.t. the class of models $\mathtt{CM}$, that is, which are true in each model from $\mathtt{CM}$. The set $\mathtt{NVal}^{\mathtt{CM}}$, in turn, comprises all the remaining wffs, that is, wffs which are not valid w.r.t. the class of models $\mathtt{CM}$. However, the set $\mathtt{NVal}^{\mathtt{M}}$ is far from being homogenous. It includes *inconsistent* (also called *unsatisfiable*) wffs, that is, wffs which are not true in any model from the class $\mathtt{CM}$. But it also includes wffs which are consistent (or *satisfiable*) without being valid, i.e. wffs which are true in some model(s) belonging to the class $\mathtt{CM}$, but are not true in other models from the class. Following a philosophical rather than a logical tradition, let us call these wffs *contingent*. To be more precise, when a class of models $\mathtt{CM}$ is fixed, the set $\mathtt{NVal}^{\mathtt{CM}}$ splits into the set $\mathtt{Inc}^{\mathtt{CM}}$

of CM-*inconsistent* wffs (i.e. wffs which are not true in any model from M) and the set $\mathtt{Ctg}^{\mathtt{CM}}$ of CM-*contingent* wffs, that is, wffs which are neither CM-valid nor CM-inconsistent.

Looking from the proof-theoretic point of view, the main challenge for a logician is to build a calculus which makes provable all the valid (w.r.t. a given class of models) wffs and only them. Sometimes, as a by-product, a calculus gives an account of inconsistent wffs as well. Analytic tableaux are paradigmatic examples here. However, contingent wffs remain beyond the scope of interest. The advocates of refutation methods see the goal differently: they aim at proof-theoretic accounts of non-validities (cf., e.g., [45], [42], [43]). But the class of non-validities includes both inconsistent wffs and contingent wffs. This distinction seems to play no role in refutation calculi, however. Last but not least, logical calculi focussed on validities and these focussed on non-validities operate with diverse formal means.

In this chapter I present a calculus which, on the one hand, differentiates between proofs of valid wffs, refutations of inconsistent wffs, and refutations of contingent wffs. On the other hand, the calculus offers a uniform proof-mechanism. This is achieved by the introduction of a kind of conceptual unifier, namely the notion of *holistically inconsistent set of wffs.* The system "calculates" such sets or, more precisely, sequents based on them. Since valid, inconsistent, and contingent wffs correspond to different, yet strictly defined, holistically inconsistent sets, a proof of a sequent based on a set of a given kind can be regarded, depending on a case, as a proof or as a refutation of the corresponding wff.

3.2 The Logical Basis

We remain at the level of Classical Propositional Calculus (CPL for short). As for the language of (the analysed version of) CPL, we assume that the vocabulary comprises a countably infinite set of propositional variables, the connectives: $\neg, \vee, \wedge, \rightarrow, \equiv$, and brackets. *Well-formed formulas* (henceforth: wffs) of the language are defined as usual. We use A, B, C, D, with subscripts when needed, as metalanguage variables for wffs, and X, Y, with or without subscripts or superscripts, as metalanguage variables for sets of wffs. The letters p, q, r, s, t are exemplary elements of the set of propositional variables of the language.

By *literals* we mean propositional variables and negations of propositional variables. Two literals are *complementary* if one of them is the negation of the other. A wff is an *elementary conjunction* iff it is a literal or is a conjunction of literals. A wff is in the *disjunctive normal form* (hereafter: DNF) iff the wff is an elementary conjunction or is a disjunction of elementary conjunctions.

Let **1** stand for truth and **0** for falsity.

Definition 3.1 (CPL-valuation; valuation). *A* CPL*-valuation is a function, v, from the set of wffs of the language of* CPL *to the set* $\{\mathbf{1}, \mathbf{0}\}$*, satisfying the following conditions:*

1. $v(\neg A) = \mathbf{1}$ *iff* $v(A) = \mathbf{0}$,
2. $v(A \vee B) = \mathbf{1}$ *iff* $v(A) = \mathbf{1}$ *or* $v(B) = \mathbf{1}$,
3. $v(A \wedge B) = \mathbf{1}$ *iff* $v(A) = \mathbf{1}$ *and* $v(B) = \mathbf{1}$,
4. $v(A \to B) = \mathbf{1}$ *iff* $v(A) = \mathbf{0}$ *or* $v(B) = \mathbf{1}$,
5. $v(A \equiv B) = \mathbf{1}$ *iff* $v(A) = v(B)$.

For brevity, in what follows we will be omitting references to CPL. By wffs we will mean wffs of the language of CPL, and by valuations we will mean CPL-valuations.

Definition 3.2 (Consistency, inconsistency, validity, and contingence w.r.t. CPL-valuations). *A set of wffs X is consistent iff there exists a valuation v such that for each* $A \in X$, $v(A) = \mathbf{1}$*; otherwise X is inconsistent. A wff B is:*

1. *consistent iff the singleton set* $\{B\}$ *is consistent,*
2. *inconsistent iff the singleton set* $\{B\}$ *is inconsistent,*
3. *valid iff for each valuation v,* $v(B) = \mathbf{1}$,
4. *contingent iff B is neither inconsistent nor valid.*

CPL-entailment, $\models$, is defined as follows:

Definition 3.3 (CPL-entailment; entailment). $X \models A$ *iff for each valuation v:*

- *if* $v(B) = \mathbf{1}$ *for every* $B \in X$*, then* $v(A) = \mathbf{1}$.

The next definition introduces the crucial notion.

Definition 3.4 (Holistically inconsistent set; HI-set). *A set of wffs X is holistically inconsistent iff X is inconsistent, but each wff in X is consistent.*

Observe that each HI-set has at least two elements.

The following are true:

Corollary 3.1. *A wff C is contingent iff $\{C, \neg C\}$ is a HI-set.*

Proof. ($\Rightarrow$) If C is a contingent wff, then there are valuations v, v^* such that $v(C) = \mathbf{1}$ and $v^*(C) = \mathbf{0}$. So both C and $\neg C$ are consistent wffs. On the other hand, the set $\{C, \neg C\}$ is inconsistent. Therefore $\{C, \neg C\}$ is a HI-set.
($\Leftarrow$) If $\{C, \neg C\}$ is a HI-set, then both C and $\neg C$ are consistent wffs. Thus C is as a contingent wff. □

Corollary 3.2. *A wff C is inconsistent iff $\{C \vee p, C \vee \neg p\}$ is a HI-set.*

Proof. ($\Rightarrow$) Assume that C is an inconsistent wff. Each of the wffs: $C \vee p$, $C \vee \neg p$, is consistent, however. On the other hand, the set $\{C \vee p, C \vee \neg p\}$ is inconsistent and hence is a HI-set.
($\Leftarrow$) If $\{C \vee p, C \vee \neg p\}$ is a HI-set, it is an inconsistent set and hence $\{C \vee p, C \vee \neg p\} \models p \wedge \neg p$. It follows that $\{C, C \vee \neg p\} \models p \wedge \neg p$ and therefore $C \models p \wedge \neg p$. Thus C is inconsistent. □

Corollary 3.3. *A wff C is valid iff $\{\neg C \vee p, \neg C \vee \neg p\}$ is a HI-set.*

Proof. ($\Rightarrow$) If C is valid, then $\neg C$ is inconsistent. But both $\neg C \vee p$ and $\neg C \vee \neg p$ are consistent wffs, and the set $\{\neg C \vee p, \neg C \vee \neg p\}$ is inconsistent. Therefore $\{\neg C \vee p, \neg C \vee \neg p\}$ is a HI-set.
($\Leftarrow$) The set $\{\neg C \vee p, \neg C \vee \neg p\}$, as a HI-set, is inconsistent. Thus $\{\neg C \vee p, \neg C \vee \neg p\} \models p \wedge \neg p$ and therefore $\neg C \models p \wedge \neg p$. It follows that $\neg C$ is an inconsistent wff and hence C is a valid wff. □

Thus validity, inconsistency and contingency of wffs are expressible in terms of HI-sets.

3.3 The System $\mathsf{HI}^{\mathsf{CPL}}$

3.3.1 Axioms, Rules, Proofs, and Refutations

Since the system we are going to present "calculates" HI-sets of CPL-wffs, we label it by $\mathsf{HI}^{\mathsf{CPL}}$.

We operate with *sequents* of the form $Y \vdash$, where Y is an at least two-element finite set of CPL-wffs. In practice, we write down a sequent $Y \vdash$ by listing the elements of Y left to the turnstile.

Definition 3.5 (CPL-axioms of $\mathsf{HI}^{\mathsf{CPL}}$)**.** *By* CPL*-axioms of* $\mathsf{HI}^{\mathsf{CPL}}$ *we mean all the wffs which are theorems of* CPL.

We designate the set of CPL-axioms of the system $\mathsf{HI}^{\mathsf{CPL}}$ by $\mathsf{Ax}_{\mathsf{CPL}}^{\mathsf{HI}^{\mathsf{CPL}}}$.

Definition 3.6 (Specific axioms of $\mathsf{HI}^{\mathsf{CPL}}$)**.** *A specific axiom of* $\mathsf{HI}^{\mathsf{CPL}}$ *is a sequent* $Y \vdash$ *such that each* $B \in Y$ *is an elementary conjunction, a conjunction of all the wffs in* Y *involves complementary literals, and no* $B \in Y$ *involves complementary literals.*

Here are examples of specific axioms:

$$p, \neg p \vdash \tag{3.1}$$

$$\neg p \wedge \neg q, p \ \vdash \tag{3.2}$$

$$\neg p \wedge \neg q, q \ \vdash \tag{3.3}$$

$$\neg p \wedge \neg q, p \wedge \neg q, q \wedge \neg p \ \vdash \tag{3.4}$$

There are only two primary *inference rules* of $\mathsf{HI}^{\mathsf{CPL}}$, namely:

$$\frac{Y \cup \{A\} \vdash \qquad Y \cup \{B\} \vdash}{Y \cup \{A \vee B\} \vdash} \tag{R_1}$$

$$\frac{Y \cup \{A\} \vdash}{Y \cup \{B\} \vdash} \quad \textit{where } (A \equiv B) \in \mathsf{Ax}_{\mathsf{CPL}}^{\mathsf{HI}^{\mathsf{CPL}}}. \tag{R_2}$$

Definition 3.7 (Proof of a sequent)**.** *A proof of a sequent* $Y \vdash$ *in* $\mathsf{HI}^{\mathsf{CPL}}$ *is a finite labelled tree regulated by the rules of* $\mathsf{HI}^{\mathsf{CPL}}$*, where the leaves are labelled with specific axioms and* $Y \vdash$ *labels the root.*

A sequent $Y \vdash$ *is provable in* $\mathsf{HI}^{\mathsf{CPL}}$ *iff the sequent* $Y \vdash$ *has at least one proof in* $\mathsf{HI}^{\mathsf{CPL}}$*.*

Here are examples of proofs:

Example 3.8. A proof of the sequent $\neg(p \vee q), \neg(\neg p \wedge \neg q) \ \vdash$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg p \wedge \neg q, p \ \vdash}{\neg p \wedge \neg q, \neg\neg p \ \vdash}\,\mathsf{R}_2 \qquad \dfrac{\neg p \wedge \neg q, q \ \vdash}{\neg p \wedge \neg q, \neg\neg q \ \vdash}\,\mathsf{R}_2}{\neg p \wedge \neg q, \neg\neg p \vee \neg\neg q \ \vdash}\,\mathsf{R}_1}{\neg p \wedge \neg q, \neg(\neg p \wedge \neg q) \ \vdash}\,\mathsf{R}_2}{\neg(p \vee q), \neg(\neg p \wedge \neg q) \ \vdash}\,\mathsf{R}_2$$

Example 3.9. A proof of the sequent $(p \vee q) \wedge \neg p, (p \vee q) \wedge \neg q, \neg p \wedge \neg q \vdash$:

$$\dfrac{\dfrac{\dfrac{\dfrac{q \wedge \neg p, p \wedge \neg q, \neg p \wedge \neg q \vdash}{q \wedge \neg p, (q \wedge \neg q) \vee (p \wedge \neg q), \neg p \wedge \neg q \vdash}\, \mathsf{R}_2}{(q \wedge \neg p), (p \vee q) \wedge \neg q, \neg p \wedge \neg q \vdash}\, \mathsf{R}_2}{(q \wedge \neg p) \vee (p \wedge \neg p), (p \vee q) \wedge \neg q, \neg p \wedge \neg q \vdash}\, \mathsf{R}_2}{(p \vee q) \wedge \neg p, (p \vee q) \wedge \neg q, \neg p \wedge \neg q \vdash}\, \mathsf{R}_2$$

Provability of a sequent $Y \vdash$ yields that Y is HI-set. Thus is due to:

Theorem 3.1 (Soundness of $\mathsf{HI}^{\mathsf{CPL}}$ w.r.t. HI-sets)**.** *Let Y be an at least two element finite set of wffs. If the sequent $Y \vdash$ is provable in $\mathsf{HI}^{\mathsf{CPL}}$, then Y is a* HI*-set.*

Proof. Clearly, if $Y \vdash$ is a specific axiom of $\mathsf{HI}^{\mathsf{CPL}}$, then Y is a HI-set.

Assume that $Y \cup \{A\}$ and $Y \cup \{B\}$ are HI-sets. Thus each wff in Y is consistent. Moreover, the set $Y \cup \{A \vee B\}$ is inconsistent – otherwise $Y \cup \{A\}$ would be consistent or $Y \cup \{B\}$ would be consistent. Suppose that the set $Y \cup \{A \vee B\}$ contains an inconsistent wff. Since each wff in Y is consistent, it follows that $A \vee B$ is inconsistent, and hence both A and B are inconsistent. But in this case neither $Y \cup \{A\}$ nor $Y \cup \{B\}$ is a HI-set. A contradiction.

It is obvious that if $X \cup \{A\}$ is a HI-set and $(A \equiv B)$ is a theorem of CPL, then $X \cup \{B\}$ is a HI-set. □

Theorem 3.1 together with corollaries 3.3, 3.2 and 3.1 yield:

Theorem 3.2.

1. *If the sequent:*

$$\neg C \vee p, \neg C \vee \neg p \vdash$$

is provable in $\mathsf{HI}^{\mathsf{CPL}}$, then C is a valid wff.

2. *If the sequent:*

$$C \vee p, C \vee \neg p \vdash$$

is provable in $\mathsf{HI}^{\mathsf{CPL}}$, then C is an inconsistent wff.

3. *If the sequent:*

$$C, \neg C \vdash$$

is provable in $\mathsf{HI}^{\mathsf{CPL}}$, then C is a contingent wff.

The next step is a non-standard one. We define provability of a wff in terms of provability of a sequent of a strictly defined form. But, contrary to what is usually done, we *do not* construe the provability of a wff C as the provability of the sequent based on C or the negation of C only. The definition runs as follows:

Definition 3.8 (Proof of a wff). *A $\mathsf{HI}^{\mathsf{CPL}}$-proof of a wff C is a proof of the sequent $\neg C \vee p, \neg C \vee \neg p \vdash$ in $\mathsf{HI}^{\mathsf{CPL}}$.*

Example 3.10. A proof of $p \to p$:

$$\dfrac{\dfrac{\dfrac{\dfrac{p, \neg p \vdash}{(p \wedge \neg p) \vee p, \neg p \vdash}\,\mathsf{R}_2}{(p \wedge \neg p) \vee p, (p \wedge \neg p) \vee \neg p \vdash}\,\mathsf{R}_2}{\neg(p \to p) \vee p, (p \wedge \neg p) \vee \neg p \vdash}\,\mathsf{R}_2}{\neg(p \to p) \vee p, \neg(p \to p) \vee \neg p \vdash}\,\mathsf{R}_2$$

Similarly, we define refutability in terms of provability of sequents of strictly defined form. This time, however, we introduce two concepts.

Definition 3.9 (Refutation[1] of a wff). *A $\mathsf{HI}^{\mathsf{CPL}}$-refutation[1] of a wff C is a proof of the sequent $C \vee p, C \vee \neg p \vdash$ in $\mathsf{HI}^{\mathsf{CPL}}$.*

Definition 3.10 (Refutation[2] of a wff). *A $\mathsf{HI}^{\mathsf{CPL}}$-refutation[2] of a wff C is a proof of the sequent $C, \neg C \vdash$ in $\mathsf{HI}^{\mathsf{CPL}}$.*

Example 3.11. A refutation[1] of $\neg(p \to p)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{p, \neg p \vdash}{\neg(\neg p \vee p) \vee p, \neg p \vdash}\,\mathsf{R}_2}{\neg(p \to p) \vee p, \neg p \vdash}\,\mathsf{R}_2}{\neg(p \to p) \vee p, \neg(\neg p \vee p) \vee \neg p \vdash}\,\mathsf{R}_2}{\neg(p \to p) \vee p, \neg(p \to p) \vee \neg p \vdash}\,\mathsf{R}_2$$

Example 3.12. A refutation[2] of $p \to q$:

$$\dfrac{\dfrac{\dfrac{\neg p, p \wedge \neg q \vdash \qquad q, p \wedge \neg q \vdash}{\neg p \vee q, p \wedge \neg q \vdash}\,\mathsf{R}_1}{p \to q, p \wedge \neg q \vdash}\,\mathsf{R}_2}{p \to q, \neg(p \to q) \vdash}\,\mathsf{R}_2$$

The following holds:

Corollary 3.4.

1. *If C has a $\mathsf{HI}^{\mathsf{CPL}}$-proof, then C is valid.*
2. *If C has a $\mathsf{HI}^{\mathsf{CPL}}$-refutation^1, then C is inconsistent.*
3. *If C has a $\mathsf{HI}^{\mathsf{CPL}}$-refutation^2, then C is contingent.*

Proof. Immediately from Theorem 3.2 and definitions 3.8, 3.9, and 3.10. □

3.3.2 The Completeness Issue

The system $\mathsf{HI}^{\mathsf{CPL}}$ is complete with respect to finite HI-sets.

A technical concept is needed.

Definition 3.11 (Normal form of a sequent)**.** *A sequent, $Y \vdash$, is in the normal form iff each $C \in Y$ is in the disjunctive normal form.*

Theorem 3.3 (Completeness of $\mathsf{HI}^{\mathsf{CPL}}$ w.r.t. HI-sets)**.** *Let Y be an at least two element finite set of wffs. If Y is a HI-set, then a sequent of the form $Y \vdash$ is provable in $\mathsf{HI}^{\mathsf{CPL}}$.*

Proof. Assume that $Y \vdash$ is in the normal form. Thus all the wffs in Y are in DNF.

By the *rank* of a sequent $Y \vdash$ (in symbols: $\mathbf{r}(Y \vdash)$) we mean the number of occurrences of the disjunction connective, $\vee$, in the wffs of Y.

Assume that Y is a finite HI-set. Note that Y has at least two elements.

Suppose that $\mathbf{r}(Y \vdash) = 0$. In this case, $Y \vdash$ is a specific axiom of the system.

Suppose that $\mathbf{r}(Y \vdash) > 0$. Let $\mathbf{r}(Y \vdash) = n$.

Inductive hypothesis. If $\mathbf{r}(X \vdash) < n$ and X is a HI-set of wffs in DNF, then the sequent $X \vdash$ is provable in $\mathsf{HI}^{\mathsf{CPL}}$.

If $\mathbf{r}(Y \vdash) = n$, where $n > 0$, the sequent $Y \vdash$ can be displayed as:

$$A_1, \ldots, A_{j-1}, B_1 \vee \ldots \vee B_k, A_{j+1}, \ldots, A_m \ \vdash$$

where $B_1, \ldots, B_k$ are elementary conjunctions and $k > 1$. As Y is a HI-set, at least one of $B_1, \ldots, B_k$ is consistent.

Let B_i be a consistent element of $\{B_1, \ldots, B_k\}$. Consider the sets Y' and Y'' defined by:

$$Y' = \{A_1, \ldots, A_{j-1}, B_i, A_{j+1}, \ldots, A_m\}$$
$$Y'' = \{A_1, \ldots, A_{j-1}, B_1 \vee \ldots \vee B_{i-1} \vee B_{i+1} \vee \ldots \vee B_k, A_{j+1}, \ldots, A_m\}$$

Clearly, $\mathbf{r}(Y' \vdash) < n$ and $\mathbf{r}(Y'' \vdash) < n$. Both $Y' \vdash$ and $Y'' \vdash$ are in the normal form.

If Y is a HI-set, so is Y'. Thus, by the inductive hypothesis, the sequent $Y' \vdash$ is provable.

As for the sequent $Y'' \vdash$, there are two cases to be considered.

Case 1. $B_1 \vee \ldots \vee B_{i-1} \vee B_{i+1} \vee \ldots \vee B_k$ is consistent. Thus Y'' is a HI-set. Hence, by the inductive hypothesis, the sequent $Y'' \vdash$ is provable. But one can get $Y \vdash$ from $Y' \vdash$ and $Y'' \vdash$ by an application of rule R_1 and then, if necessary, of rule R_2.

Case 2. $B_1 \vee \ldots \vee B_{i-1} \vee B_{i+1} \vee \ldots \vee B_k$ is inconsistent. Thus all the disjuncts (of the just considered disjunction) are inconsistent. If follows that B_i is CPL-equivalent to A_j. (Clearly we have $B_i \models A_j$. But, as all the disjuncts of $B_1 \vee \ldots \vee B_{i-1} \vee B_{i+1} \vee \ldots \vee B_k$ are inconsistent, their negations are valid and hence from $A_j \models B_1 \vee \ldots \vee B_k$ we get $A_j \models B_i$.) Thus one can get $Y \vdash$ from $Y' \vdash$ by R_2.

Now assume that $Y \vdash$ is not in the normal form. In order to complete the proof it suffices to observe that each CPL-wff is CPL-equivalent to a wff in DNF and thus one can always reach a wff from its DNF-counterpart by applying rule R_2. □

As a consequence of Theorem 3.3, Corollary 3.4, and definitions 3.8, 3.9, 3.10 one gets:

Theorem 3.4.

1. *A wff C is valid iff C has a* $\mathsf{HI}^{\mathsf{CPL}}$*-proof.*
2. *A wff C is inconsistent iff C has a* $\mathsf{HI}^{\mathsf{CPL}}$*-refutation*^{1}.
3. *A wff C is contingent iff C has a* $\mathsf{HI}^{\mathsf{CPL}}$*-refutation*^{2}.

3.4 Final Remarks

The methodology used in the construction of the system $\mathsf{HI}^{\mathsf{CPL}}$, and the basic idea of the completeness proof, are very much alike to the methodology and idea applied, for different purposes, in [44].

As for this chapter, the homogeneity effect has been achieved by using the notion of HI-set as a conceptual unifier. It is worth to note that the concept of minimally inconsistent set could have been used for this purpose as well. A set of wffs X is a minimally inconsistent set (MI-set for short) iff X is inconsistent, but each proper subset of X is consistent. When one deals with CPL, inconsistency, validity and contingency of wffs are expressible in terms of MI-sets as follows:

- A wff C is inconsistent iff $\{C\}$ is a MI-set.
- A wff C is valid iff $\{\neg C\}$ is a MI-set.
- A wff C is contingent iff $\{C, \neg C\}$ is a MI-set.

Thus once we have a system which "calculates" MI-sets, we get an alternative solution. A system of this kind already exists (cf. [65]) and will be presented in Chapter 10 of this book. Its applications to the case considered are pointed out in section 11.3 of Chapter 11. The pros and cons issue remains to be studied.

The last remark is this. As for Classical Logic and some non-classical logics, one can define entailment by the clause:

(#) *X entails A iff the set $X \cup \{\neg A\}$ is inconsistent.*

However, a set of wffs can be inconsistent in different ways. One can differentiate between holistic inconsistency, minimal inconsistency, plain inconsistency, and so forth. Given this, one can then define different kinds of entailment, depending on the kind of inconsistency involved. In particular, if "inconsistent" were replaced in (#) above with "holistically inconsistent," we would get a non-Tarskian consequence relation with interesting properties. The system $\mathsf{HI}^{\mathsf{CPL}}$ offers a proof-theoretic account of entailment defined in this way (for the classical propositional case). However, this is another story.

Part II

Epistemizing

Chapter 4

Three Problems Concerning Knowledge and Belief

4.1 Introduction

Most philosophers agree that knowledge is not just true belief: it is either true belief "plus something else" or true belief of some special kind. The standard tripartite analysis of "knowing that" conceives knowledge as true justified belief: one knows that p just in case p holds/is true, one believes that p, and one is justified in one's believing that p. After the famous Gettier's ([15]) paper, however, it has become clear that the standard analysis is too broad, as it does not cope with counterexamples presented first by him and then by others. We know that the "something else" clause cannot be just being justified in believing. So the problem has arisen: how to improve and/or supplement the third clause of the standard tripartite analysis (and possibly the other clause(s) too) in order to cope with Gettier-style counterexamples. Numerous solutions were proposed, but no consensus has been reached yet.[11]

In this chapter I show that the "true belief plus something else" account of propositional knowledge faces problems which reach beyond the Gettier problem. When analysing the account, I leave the "something else" part unspecified. This is a deliberate tactics, as the problems would

[11] The story is widely known to philosophers (or at least should be), but not necessarily so to logicians. An interested reader may consult, e.g., [41], or [22]. In the last years the interest in the problem seems to decrease. It is even claimed that the problem cannot have a satisfactory solution (cf., e.g., Floridi's [12] paper).

not disappear even if the "something else" clause were specified in a way that the Gettier-type counterexamples would be blocked. I address these problems in sections 4.4 and 4.5. They are due to two paradoxes which, to my knowledge, have not been noticed yet. I coin them the Paradox of Astounding Consequent and the Paradox of Doxastic Agency. Both hold regardless of how the missing clause of the "true belief plus something else" account of propositional knowledge is specified. Section 4.6, in turn, is devoted to an issue which emerges when propositional knowledge is conceptualized as true conviction. Again, this issue is rooted in some paradox, which I call here the Doxastic Misfortune Paradox.

4.2 The Standard Tripartite Account of Knowledge Generalized

The "true belief plus something else" account of propositional knowledge will be represented here by the following schema:

$(\mathsf{TB}^{+x}) \qquad \mathsf{K}p \equiv (p \wedge \mathsf{B}p) \wedge \Omega p$

where p is a propositional variable, K and B stand for the knowledge operator and the belief operator, respectively, and Ω is an unspecified expression by means of which the "something else" component is worded. It is neither assumed nor denied that the belief operator occurs in Ω, and similarly for other expressions and/or operators. Moreover, at the moment we remain neutral in the controversy whether B in (TB^{+x}) represents "weak" or "strong" belief.

Needless to say, leaving the Ω-clause unspecified has a price. At the formal level we are left with the possibility of a proof-theoretic analysis only.

4.3 Logical Preliminaries

We remain at the propositional level, and the basic propositional logic chosen is Classical Propositional Logic (hereafter: CPL). At the first step, the language of CPL is enriched with the knowledge operator K and the belief operator B. The concept of well-formed formula (wff for short) is defined in the standard way. We use the letters $p, q, r, s, \ldots$ for propositional variables, and the symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\equiv$ for the connectives of negation, conjunction, disjunction, implications, and equivalence, respectively. The letters A, C, D, possibly with subscripts, are metalanguage variables for wffs, and the letters X, Y are metalanguage variables for sets of wffs. At the second step, we assume that the syntax of the language is extended to the effect that expressions of the form ΩA are wffs and, as such, can be constituents of complex wffs.[12] No further assumptions concerning the syntactic form of Ω are made.

CPL-axioms are defined as usual. In each case considered *Modus Ponens* and (unrestricted) Uniform Substitution are primary inference rules. This allows us to apply all the derivable CPL-rules. Assumptions and rules pertaining to epistemic operators will be specified in due course.

4.3.1 Notional and Attitudinal Readings of Epistemic Operators

A formula of the form $\mathsf{K}A$ can be read either as: "it is known that A" (in short: A is known), or as: "an agent knows that A." Observe that under the first reading (let me call it "notional"), formula $\mathsf{K}A$ speaks about *knowledge* and says that A is an item of knowledge. When the second reading comes into play, formula $\mathsf{K}A$ speaks about an agent's *attitude* towards A, identifying this attitude as knowledge-that. A formula of the form $\mathsf{B}A$ can be read and analysed along similar lines: either notionally, as: "A is a subject of belief," or attitudinally, as: "an agent believes that A." Since we are primarily interested here in relations between concepts and only secondarily between agent's attitudes, the notional readings of K and B will be adopted below by default unless stated otherwise.

When one works with alethic modal propositional logic based on CPL, the possibility operator, $\Diamond$, can be defined in terms of the necessity operator, $\Box$, by the equivalence:

$$\Diamond A \equiv \neg\Box\neg A \tag{4.1}$$

By analogy, the following formula:

$$\neg\mathsf{K}\neg A \tag{4.2}$$

can be regarded as expressing *epistemic possibility* of A. This is, however, a very specific and rather weak concept of epistemic possibility: an epistemic possibility of a proposition amounts to the fact that the negation of the proposition does not constitute an item of knowledge.[13]

[12] More formally, the following clause is added to the definition of wffs: (*) if A is a wff, then ΩA is a wff.

[13] An attitudinal account of formula (4.2) usually includes a reference to all that is known by an agent (see, e.g., [21], p. 3.). This reference is lacking in our case.

4.4 The First Problem

The first problem of the "true belief plus something else" account of propositional knowledge I am going to point out here, is:

(I) *For an epistemically possible proposition to hold, it suffices that its classical negation is believed and satisfies the third clause of the definition of knowledge, whatever this clause occurs to be.*

4.4.1 The Paradox of Astounding Consequent

The problem arises due to the fact that one can derive from formula $\mathtt{TB}^{+x}$, the formula:

$$(\mathtt{PAC}) \qquad \neg\mathsf{K}\neg p \wedge (\mathsf{B}\neg p \wedge \Omega\neg p) \to p$$

by using CPL-means only. Here is a derivation:[14]

a.1.	$\mathsf{K}p \equiv (p \wedge \mathsf{B}p) \wedge \Omega p$	($\mathtt{TB}^{+x}$)
a.2.	$(p \wedge \mathsf{B}p) \wedge \Omega p \to \mathsf{K}p$	(a.1. CPL)
a.3.	$p \wedge (\mathsf{B}p \wedge \Omega p) \to \mathsf{K}p$	(a.2. CPL)
a.4.	$p \to (\mathsf{B}p \wedge \Omega p \to \mathsf{K}p)$	(a.3. CPL)
a.5.	$\neg p \to (\mathsf{B}\neg p \wedge \Omega\neg p \to \mathsf{K}\neg p)$	(a.4. $p/\neg p$)
a.6.	$\neg(\mathsf{B}\neg p \wedge \Omega\neg p \to \mathsf{K}\neg p) \to p$	(a.5. CPL)
a.7.	$(\mathsf{B}\neg p \wedge \Omega\neg p) \wedge \neg\mathsf{K}\neg p \to p$	(a.6. CPL)
(PAC)	$\neg\mathsf{K}\neg p \wedge (\mathsf{B}\neg p \wedge \Omega\neg p) \to p$	(a.7. CPL)

Recall that formula $\neg\mathsf{K}\neg p$ reads: "p is epistemically possible," which amounts to: "$\neg p$ does not constitute an item of knowledge." Formula $\mathsf{B}\neg p$ may read, in turn: "$\neg p$ is a subject of belief" or simply "it is believed that $\neg p$." The reading of $\Omega\neg p$ remains unspecified, but its function is clear: this is the *negation* of p that satisfies the "something else" clause of the analysed definition of knowledge. So formula PAC says something like: p is the case if p is epistemically possible, although this is the

[14] The left column comprises numbers of lines of the derivation, the middle column is the derivation itself, while the items of the right column describe where does the formula of a line come from. When 'CPL' occurs in the right column together with number(s) of line(s), it means that CPL-based rule (primary or derivable) has been applied. CPL-rules used are not specified, since the transformations performed are simple enough to make visible what rule has been applied at a given step. When Uniform Substitution had been applied, the item of the third column informs what was substituted for what.

negation of p that is believed and satisfies the missing clause of the definition of propositional knowledge. Or, to put it differently, if the negation of p is not known, but nevertheless is believed and satisfies the missing third clause, then – surprisingly – p is the case. Regardless of the reading chosen, we arrive at an astounding consequence. In what follows, formula PAC will be referred to as the *Paradox of Astounding Consequent*.

Observe that the above derivation of PAC employs CPL-means only and does not rely on any specific assumptions concerning the underlying logics of knowledge and belief.

4.4.2 The Issue of Consistency of the Antecedent of PAC

A deductive inference ensures the transmission of truth: if only all the premises are simultaneously true, so is the conclusion. It happens, however, that an inference is deductive although its premises cannot be simultaneously true. An inference from a proposition and its negation to *any* proposition is deductive in view of Classical Logic and, as such, ensures the transmission of truth from premises to the conclusion. Yet, it does it for a tricky reason: the premises cannot be simultaneously true. So the transmission of truth principle is not violated, but the truth of the conclusion is not ensured. If the underlying logic is not paraconsistent, each inference from an inconsistent (in view of the logic) set of premises is deductive, but its deductiveness does not provide a warranty of the truth of the conclusion. Thus when we reason to the consequent of an implication whose antecedent is inconsistent, and this is the implication that makes our inference deductive, we do not provide an argument in favour of the truth of the conclusion. So the question arises: *is the antecedent of* PAC *consistent*?

Clearly, what one would like to have is consistency in view of the underlying logic of K, B, and Ω. Yet, since Ω has been left unspecified, no semantic analysis is possible. What is possible, however, is a sketchy proof-theoretic analysis. To this end, it is convenient to identify a logic with the derivability relation determined by its axioms and primary inference rules.

Suppose that we have a formal language that satisfies the conditions specified in section 4.3 above. Let Ax_L be a non-empty set of wffs of the language that contains all the CPL-axioms and possibly some other formulas. Elements of Ax_L will be called *L-axioms*. Let Π_L be a set of primary inference rules whose elements are *Modus Ponens*, Uniform Substitution, and possibly some other rules, called *specific L-rules*.

The expression $X \vdash_L A$ reads "A is *L-derivable from X*." $X \vdash_L A$ holds iff there exists at least one derivation of A from $X \cup Ax_L$ such that the rules employed belong to Π_L. Thus $\vdash_L A$ claims that A can be derived from L-axioms by means of inference rules belonging to Π_L, and thus reads: "A is a theorem of L." Let $\bot$ abbreviate $p \wedge \neg p$. We say that a set of wffs X is *L-inconsistent* iff $X \vdash_L \bot$ holds; otherwise X is *L-consistent*. A wff A is *L-inconsistent* iff the set $\{A\}$ is, and similarly for L-consistency. A wff A is called *L-refuted* iff $\nvdash_L A$, that is, A is not a theorem of L.[15]

Having all these auxiliary concepts at hand, we can now reduce the consistency issue of the antecedent of `PAC`, that is, of the formula:

$$\neg \mathsf{K} \neg p \wedge (\mathsf{B} \neg p \wedge \Omega \neg p) \tag{4.3}$$

to questions of the form: 'What property the underlying logic L *must not* have in order to ensure L-consistency of formula (4.3)?"

There are many possible yet correct answers to this general question. For example, assume that the relation $\vdash_L$ satisfies the following condition:

(♣) if $X \cup \{A\} \vdash_L \bot$, then $X \vdash_L A \to \bot$.

and that L is closed under the following rule of elimination of double negations:[16]

$$\frac{A[\neg\neg C]}{A[C]} \qquad (E_{\neg\neg})$$

which allows for the replacement of each occurrence of $\neg\neg C$ in A with an occurrence of C.

Now the answer is: (4.3) is L-consistent if the following formula

$$\mathsf{B}p \wedge \Omega p \to \mathsf{K}p \tag{4.4}$$

is L-refuted, that is, is not a theorem of L. For assume that formula (4.3) is L-inconsistent, that is, the following holds:

$$\neg \mathsf{K} \neg p \wedge (\mathsf{B} \neg p \wedge \Omega \neg p) \vdash_L \bot$$

[15] Observe that this category includes *also* wffs which are *L-contingent*, that is, are neither L-inconsistent nor L-derivable.

[16] This assumption, as well as the previous one, is not trivial, as Ω has been left unspecified.

Thus, by (♣):

$$\vdash_L \neg\mathsf{K}\neg p \wedge (\mathsf{B}\neg p \wedge \Omega\neg p) \rightarrow \bot$$

Therefore, as $\vdash_{\mathsf{CPL}} \subseteq \vdash_L$, we have:

$$\vdash_L \neg(\mathsf{B}\neg p \wedge \Omega\neg p \rightarrow \mathsf{K}\neg p) \rightarrow \bot$$

and hence:

$$\vdash_L \neg\bot \rightarrow (\mathsf{B}\neg p \wedge \Omega\neg p \rightarrow \mathsf{K}\neg p)$$

Since $\vdash_L \neg\bot$ holds (recall that $\bot$ abbreviates $p \wedge \neg p$), we get:

$$\vdash_L \mathsf{B}\neg p \wedge \Omega\neg p \rightarrow \mathsf{K}\neg p$$

But L is closed under Uniform Substitution. Thus we also have:

$$\vdash_L \mathsf{B}\neg\neg p \wedge \Omega\neg\neg p \rightarrow \mathsf{K}\neg\neg p$$

By assumption, L is closed under the rule $E_{\neg\neg}$ of elimination of double negations. Hence:

$$\vdash_L \mathsf{B}p \wedge \Omega p \rightarrow \mathsf{K}p$$

It follows that if formula $\mathsf{B}p \wedge \Omega p \rightarrow \mathsf{K}p$ is not a theorem of L, but L is closed under rule $E_{\neg\neg}$ and the condition (♣) is satisfied w.r.t. L, then the antecedent of `PAC`, that is, the formula:

$$\neg\mathsf{K}\neg p \wedge (\mathsf{B}\neg p \wedge \Omega\neg p)$$

is L-consistent.

By assumption, $\vdash_{\mathsf{CPL}} \subseteq \vdash_L$. Hence if the formulas:

$$\mathsf{B}p \rightarrow \mathsf{K}p \tag{4.5}$$

$$\Omega p \rightarrow \mathsf{K}p \tag{4.6}$$

were theorems of L, formula (4.4) would be a theorem of L. Thus when (4.4) is L-refuted (which, in the current setting, ensures L-consistency of the antecedent of `PAC`), formula (4.5) is L-refuted or formula (4.6) is L-refuted. But if (4.5) is L-refuted, so is:

$$\mathsf{K}p \equiv \mathsf{B}p \tag{4.7}$$

and if (4.6) is L-refuted, the following formula:

$$\mathsf{K}p \equiv \Omega p \tag{4.8}$$

is L-refuted as well. Hence the underlying logic that ensures the consistency of formula (4.4) does not reduce knowledge to belief or does not identify knowledge with the Ω-clause of the definition $\mathtt{TB}^+$, whatever the clause is. On the other hand, if (4.7) were a theorem of L, formula (4.4) would be L-inconsistent, and similarly for (4.8). In both cases PAC would be "neutralized" in the sense that a deductive inference based on it and leading to p is not an argument in favour of the truth of p. However, this is a kind of Pyrrhic victory: K collapses into B or into Ω. Certainly, no philosopher who subscribes to a tripartite account of propositional knowledge would be happy with this.

Remark 4.1. Observe that if formula (4.4) is a theorem of L and neither B nor Ω is supposed to be factive (i.e. $\mathsf{B}p \to p$ and $\Omega p \to p$ are L-refuted), then PAC is neutralized, but it can happen that a false proposition, A, is simultaneously known, since $\mathsf{B}A \wedge \Omega A$ holds, and not known, because $\mathtt{TB}^+$ still holds.

4.4.3 Consequences for the Standard Tripartite Account

The standard, tripartite account of propositional knowledge can be represented by the following schema:

(TBJ) $\quad \mathsf{K}p \equiv (p \wedge \mathsf{B}p) \wedge \mathsf{J}p$

where J stands for the "being justified in believing" clause. Clearly, by a reasoning analogous to that justifying formula PAC, one gets:

$$\neg\mathsf{K}\neg p \wedge (\mathsf{B}\neg p \wedge \mathsf{J}\neg p) \to p \tag{4.9}$$

which is equally counter-intuitive as PAC. The following exemplary instance of formula (4.9) illustrate this point:

> *If it is not known that Martians do not exist, but it is believed, and justifiably so, that Martians do not exist, then Martians exist.*

It seems that the above considerations provide arguments against the standard, tripartite analysis of propositional knowledge which are less loaded than the Gettier-style ones.

4.5 The Second Problem

The second problem of the "true belief plus something else" account of propositional knowledge I am pointing out in this chapter, is:

(II) *Knowing that a proposition holds reduces to believing that it holds together with believing that the proposition satisfies the third clause of the definition of knowledge, whatever this clause occurs to be.*

Thus, generally speaking, the analysed account of propositional knowledge boils down to a purely doxastic account.

4.5.1 The Conceptual Setting

In contradistinction to the first problem, the second one is strongly dependent upon specific assumptions concerning the underlying logics of knowledge and belief. But before I will list them, let me pay attention to some immediate CPL-consequences of the (schematic) definition of knowledge $\mathtt{TB}^{+x}$.

From formula $\mathtt{TB}^{+x}$ one gets the epistemic version of the (alethic) modal formula **T**:

$(\mathbf{T}_{\mathsf{K}})$ $\quad \mathsf{K}p \rightarrow p$

Thus knowledge is factive/truthful. $\mathtt{TB}^{+x}$ also gives:

(KB1) $\quad \mathsf{K}p \rightarrow \mathsf{B}p$

Formula KB1 links knowledge and beliefs, making knowledge a species of belief. It frequently occurs in considerations devoted to multimodal, epistemic-doxastic logics, often (but not always) in the role of a "bridge axiom" of such a logic.

The belief operator B. As for the belief operator, the doxastic versions of modal formulas **K** and **D** will be employed as premises:

$(\mathbf{K}_{\mathsf{B}})$ $\quad \mathsf{B}(p \rightarrow q) \rightarrow (\mathsf{B}p \rightarrow \mathsf{B}q)$

$(\mathbf{D}_{\mathsf{B}})$ $\quad \mathsf{B}p \rightarrow \neg\mathsf{B}\neg p$

The following Rule of Necessitation for the belief operator is adopted:

$$\frac{A}{\mathsf{B}A} \qquad (N_{\mathsf{B}})$$

It is well-known that $\mathbf{K}_{\mathsf{B}}$, in the presence of N_{B}, makes the following formula provable:

$$\mathsf{B}(p \wedge q) \equiv \mathsf{B}p \wedge \mathsf{B}q \qquad (4.10)$$

and that the monotony rule M_{B} for the operator B:

$$\frac{A \to C}{\mathsf{B}A \to \mathsf{B}C} \tag{M_{B}}$$

is derivable in the above setting.

The knowledge operator K. In the case of the knowledge operator, the formulas:

($\mathbf{K}_{\mathsf{K}}$) $\mathsf{K}(p \to q) \to (\mathsf{K}p \to \mathsf{K}q)$

($\mathbf{B}^*_{\mathsf{K}}$) $\neg p \to \mathsf{K}\neg\mathsf{K}p$

will perform the role of premises.

In addition, the Rule of Necessitation for the knowledge operator:

$$\frac{A}{\mathsf{K}A} \tag{N_{K}}$$

will be adopted. Needless to say, the monotony rule M_{K} for the operator K:

$$\frac{A \to C}{\mathsf{K}A \to \mathsf{K}C} \tag{M_{K}}$$

is now derivable.

Formula $\mathbf{B}^*_{\mathsf{K}}$ can be read as follows: if $\neg p$ is the case, then it is known that it is not known that p. In other words, if p is not the case, then it is known that p does not constitute an item of knowledge. Viewed this way, formula $\mathbf{B}^*_{\mathsf{K}}$ complements the claim of the "factivity" or truthfulness formula $\mathbf{T}_{\mathsf{K}}$, although these formulas are proof-theoretically independent in the current setting.

As for normal modal propositional logics (which we operate with here), $\mathbf{B}^*_{\mathsf{K}}$ is proof-theoretically equivalent to any of the following formulas:

($\mathbf{B}_{\mathsf{K}}$) $p \to \mathsf{K}\neg\mathsf{K}\neg p$

($\mathbf{B}'_{\mathsf{K}}$) $\neg\mathsf{K}\neg\mathsf{K}p \to p$

which, in turn, are epistemic versions of the Brouwerian formula/axiom $\mathbf{B}$.[17]

The positive and negative introspection formulas for knowledge, that is:

[17] By the way, under the notional reading of its respective parts, formula $\mathbf{B}'_{\mathsf{K}}$ says: ($\star$) If it is epistemically possible that p is an item of knowledge, then p is true.

$(\mathbf{4}_{\mathsf{K}})$ $\quad \mathsf{K}p \to \mathsf{KK}p$

$(\mathbf{5}_{\mathsf{K}})$ $\quad \neg\mathsf{K}p \to \mathsf{K}\neg\mathsf{K}p$

are neither accepted nor rejected at the moment; no result presented in this section necessarily rely on them.

Linking formulas. We will be using as premises the following linking formulas:

(KB1) $\quad \mathsf{K}p \to \mathsf{B}p$

(KB2) $\quad \mathsf{B}p \to \mathsf{KB}p$

(BK2) $\quad \mathsf{K}p \to \mathsf{BK}p.$

Formula KB1 ensures that whatever is known, is also believed. According to KB2, if a proposition is a subject of belief, then the proposition saying this is an item of knowledge. Formula BK2 says that if a proposition is an item of knowledge, then the proposition claiming its being an item of knowledge is a subject of belief.

Clearly, KB2 and KB1 jointly make provable the following formula:

$(\mathbf{4}_{\mathsf{B}})$ $\quad \mathsf{B}p \to \mathsf{BB}p$

which expresses the so-called positive introspection of beliefs.

4.5.2 A Reduction to a Purely Doxastic Account: The Paradox of Doxastic Agency

Our second problem is rooted in the derivability of the following formula:

(PDA) $\quad \mathsf{K}p \equiv \mathsf{B}p \wedge \mathsf{B}\Omega p$

in the current setting, that is, from formula $\mathtt{TB}^{+x}$ and the set of formulas $\{\mathbf{K}_{\mathsf{B}}, \mathbf{D}_{\mathsf{B}}, \mathbf{K}_{\mathsf{K}}, \mathbf{B}^{*}_{\mathsf{K}}, \mathsf{BK2}, \mathsf{KB1}, \mathsf{KB2}\}$.

According to formula PDA, a proposition p is an item of knowledge just in case p is a subject of belief and it is believed that the Ω-condition

When $\mathbf{B}'_{\mathsf{K}}$ is a *theorem/axiom* of an epistemic logic, its claim pertain to any proposition. In this case it ensures that:

(△) Each proposition of which it is epistemically possible that it is an item of knowledge, is true.

As formulas $\mathbf{B}'_{\mathsf{K}}$ and $\mathbf{B}^{*}_{\mathsf{K}}$ are CPL-equivalent, one may conclude that the principle (△) is ensured when formula $\mathbf{B}^{*}_{\mathsf{K}}$ is a theorem of an epistemic-doxastic logic in question.

is satisfied w.r.t. p. Thus the $\mathtt{TB}^{+x}$ account of propositional knowledge reduces to a purely doxastic account. The acronym PDA refers to *Paradox of Doxastic Agency*. Formula PDA is paradoxical in its allowing to create propositional knowledge out of *false* beliefs concerning both the subject matter and the fulfilment of the Ω-clause.

An Exemplary derivation of PDA. For brevity, the derivation presented below employs the rules M_{B} and M_{K}, and uses formulas (4.10) and $\mathbf{4}_{\mathsf{B}}$ as premises. These rules and formulas, however, are derivable in the setting described in section 4.5.1.

b.1.	$\mathsf{K}p \equiv (p \wedge \mathsf{B}p) \wedge \Omega p$	($\mathtt{TB}^{+x}$)
b.2.	$\mathsf{K}p \to (p \wedge \mathsf{B}p) \wedge \Omega p$	(b.1. CPL)
b.3.	$\mathsf{BK}p \to \mathsf{B}((p \wedge \mathsf{B}p) \wedge \Omega p)$	(b.2. M_{B})
b.4.	$\mathsf{B}(p \wedge q) \equiv \mathsf{B}p \wedge \mathsf{B}q$	(4.10)
b.5.	$\mathsf{B}(p \wedge q) \to \mathsf{B}p \wedge \mathsf{B}q$	(b.4. CPL)
b.6.	$\mathsf{B}((p \wedge \mathsf{B}p) \wedge \Omega p) \to \mathsf{B}(p \wedge \mathsf{B}p) \wedge \mathsf{B}\Omega p$	(b.5. $p/(p \wedge \mathsf{B}p)$, $q/\Omega p$)
b.7.	$\mathsf{BK}p \to \mathsf{B}(p \wedge \mathsf{B}p) \wedge \mathsf{B}\Omega p$	(b.3., b.6. CPL)
b.8.	$\mathsf{B}(p \wedge \mathsf{B}p) \to \mathsf{B}p \wedge \mathsf{BB}p$	(b.5 $q/\mathsf{B}p$)
b.9.	$\mathsf{BK}p \to (\mathsf{B}p \wedge \mathsf{BB}p) \wedge \mathsf{B}\Omega p$	(b.7., b.8. CPL)
b.10.	$\mathsf{BK}p \to \mathsf{B}p \wedge \mathsf{B}\Omega p$	(b.9. CPL)
b.11.	$\mathsf{K}p \to \mathsf{BK}p$	(BK2)
b.12.	$\mathsf{K}p \to \mathsf{B}p \wedge \mathsf{B}\Omega p$	(b.11., b.10. CPL)
b.13.	$(p \wedge \mathsf{B}p) \wedge \Omega p \to \mathsf{K}p$	(b.1. CPL)
b.14.	$\mathsf{B}((p \wedge \mathsf{B}p) \wedge \Omega p) \to \mathsf{BK}p$	(b.13. M_{B})
b.15.	$\mathsf{B}p \wedge \mathsf{B}q \to \mathsf{B}(p \wedge q)$	(b.4. CPL)
b.16.	$\mathsf{B}(p \wedge \mathsf{B}p) \wedge \mathsf{B}\Omega p \to \mathsf{B}((p \wedge \mathsf{B}p) \wedge \Omega p)$	(b.15. $p/(p \wedge \mathsf{B}p), q/\Omega p$)
b.17.	$\mathsf{B}(p \wedge \mathsf{B}p) \wedge \mathsf{B}\Omega p \to \mathsf{BK}p$	(b.16., b.14. CPL)
b.18.	$\mathsf{B}p \wedge \mathsf{BB}p \to \mathsf{B}(p \wedge \mathsf{B}p)$	(b.15 $q/\mathsf{B}p$)
b.19.	$(\mathsf{B}p \wedge \mathsf{BB}p) \wedge \mathsf{B}\Omega p \to \mathsf{BK}p$	(b.17, b.18 CPL)
b.20.	$\mathsf{B}p \to \mathsf{BB}p$	($\mathbf{4}_{\mathsf{B}}$)
b.21.	$\mathsf{B}p \wedge \mathsf{B}\Omega p \to \mathsf{BK}p$	(b.20., b.19. CPL)
b.22.	$\neg p \to \mathsf{K}\neg\mathsf{K}p$	($\mathbf{B}^{*}_{\mathsf{K}}$)
b.23.	$\mathsf{K}p \to \mathsf{B}p$	(KB1)
b.24.	$\mathsf{K}\neg\mathsf{K}p \to \mathsf{B}\neg\mathsf{K}p$	(b.23. $p/\neg\mathsf{K}p$)
b.25.	$\neg p \to \mathsf{B}\neg\mathsf{K}p$	(b.22., b.24. CPL)
b.26.	$\neg\mathsf{B}\neg\mathsf{K}p \to p$	(b.25 CPL)
b.27.	$\mathsf{B}p \to \neg\mathsf{B}\neg p$	($\mathbf{D}_{\mathsf{B}}$)
b.28.	$\mathsf{BK}p \to \neg\mathsf{B}\neg\mathsf{K}p$	(b.27 $p/\mathsf{K}p$)
b.29.	$\mathsf{BK}p \to p$	(b.28., b.26. CPL)

b.30.	$\mathsf{KBK}p \to \mathsf{K}p$	(b.29. M_{K})
b.3.	$\mathsf{B}p \to \mathsf{KB}p$	(KB2)
b.32.	$\mathsf{BK}p \to \mathsf{KBK}p$	(b.31. $p/\mathsf{K}p$)
b.33.	$\mathsf{BK}p \to \mathsf{K}p$	(b.32., b.30. CPL)
b.34.	$\mathsf{B}p \wedge \mathsf{B}\Omega p \to \mathsf{K}p$	(b.21., b.33. CPL)
(PDA)	$\mathsf{K}p \equiv \mathsf{B}p \wedge \mathsf{B}\Omega p$	(b.34., b.12. CPL)

The set of premises explicitly used comprises formula $\mathtt{TB}^{+x}$ and two subsets: $\{\mathbf{D}_{\mathsf{B}}, \mathbf{B}^*_{\mathsf{K}}, \mathsf{BK2}, \mathsf{KB1}, \mathsf{KB2}\}$ and $\{(4.10), \mathbf{4}_{\mathsf{B}}\}$. Formula $\mathbf{4}_{\mathsf{B}}$ stems from formulas KB1 and KB2, while in order to have formula (4.10) one needs formula $\mathbf{K}_{\mathsf{B}}$ and the Rule of Necessitation N_{B} (which we both have). Modal inference rules explicitly used are M_{B} and M_{K}. The former is derivable once we have formula K_{B} and the Rule of Necessitation N_{B} (as we do), while the latter is derivable when we have formula $\mathbf{K}_{\mathsf{K}}$ and the Rule of Necessitation N_{K} (which we have as well). (By the way, formula KB1 is superfluous in the sense of being an immediate consequence of $\mathtt{TB}^{+x}$.) Thus one can say that PDA can be derived from formula $\mathtt{TB}^{+x}$ and the set of formulas $\{\mathbf{K}_{\mathsf{B}}, \mathbf{D}_{\mathsf{B}}, \mathbf{K}_{\mathsf{K}}, \mathbf{B}^*_{\mathsf{K}}, \mathsf{BK2}, \mathsf{KB1}, \mathsf{KB2}\}$ by using CPL-means as well as the necessitation rules N_{B} and N_{K}.

4.5.3 Andrew's Muddle

Is PDA paradoxical in the sense of allowing for contradictions? Certainly, it is not an antinomy. However, consider the following hypothetical situation. We are speaking about beliefs and these, at least at most accounts, need not be factive (one can truly believe a false proposition). Let p be a false proposition such that a cognitive agent named, say, Andrew, believes that p is the case and believes, mistakenly but still, that the Ω-clause is satisfied w.r.t. p. So the conditions $\mathsf{B}p$ and $\mathsf{B}\Omega p$ are true with regard to Andrew. By PDA we may conclude that Andrew knows that p. But the "true belief plus something else" account of knowledge is still in place, and, as from formula $\mathtt{TB}^{+}$ we get $\mathsf{K}p \to p$ and hence (in CPL) $\neg p \to \neg\mathsf{K}p$, the (assumed) falsity of p gives, by Bivalence, the truth of $\neg p$ and thus the falsity of $\mathsf{K}p$: it is not the case that Andrew knows that p. So does Andrew know that p, or not? Both. This is a conclusion which nobody – leaving dialetheism apart – may want to have.

Leaving aside our Andrew (and attitudinal readings of the respective operators too), we can summarize the situation as follows: *if a false proposition, say, p, is believed and it is mistakenly believed that the Ω-clause of a definition of propositional knowledge is satisfied with respect* *to p, then the proposition p both constitutes and item of knowledge and does not constitute an item of knowledge.*

Remark 4.2. The reasoning presented above is based on the assumption that false propositions can be subjects of belief. If we block this possibility, the problem does not arise. As it is well-known, there are epistemic-doxastic logics in which beliefs collapse to knowledge, i.e. $\mathsf{B}p \to \mathsf{K}p$ holds, and thus, assuming the factivity of knowledge, beliefs become factive as well. However, indistinguishability of beliefs and knowledge is a vice rather than a virtue of an epistemic-doxastic logic. No doubt, when we directly reduce beliefs to knowledge, the paradoxicality of PDA disappears. Such a move, however, is tantamount to depriving beliefs the basic property that, traditionally, distinguishes them from knowledge. And resolving a local conflict by dropping nuclear bombs is, for sure, not the best idea.

4.5.4 Consequences for the TBJ Account of Propositional Knowledge

It is obvious that when one operates with the standard, tripartite account of propositional knowledge (depicted by the formula TBJ), in which the Ω-clause boils down to the justification requirement, then, *ceteris paribus*, one gets:

$$\mathsf{K}p \equiv \mathsf{B}p \wedge \mathsf{BJ}p \tag{4.11}$$

But from TBJ and (4.11) one immediately gets (by CPL):

$$\mathsf{B}p \wedge \mathsf{BJ}p \equiv (p \wedge \mathsf{B}p) \wedge \mathsf{J}p \tag{4.12}$$

and this gives, int. al.:

$$\mathsf{B}p \wedge \mathsf{BJ}p \to p \tag{4.13}$$

$$\mathsf{B}p \wedge \mathsf{BJ}p \to \mathsf{J}p \tag{4.14}$$

Assuming that being justified in believing that p yields believing that p, that is, the following holds:

$$\mathsf{BJ}p \to \mathsf{B}p \tag{4.15}$$

we end with the following formulas:

$$\mathsf{BJ}p \to \mathsf{J}p \tag{4.16}$$

$$\mathsf{BJ}p \to p \tag{4.17}$$

which ensure that believing in being justified in believing yields being justified in believing, and that believing in being justified in believing warrants that what is believed, is the case.[18] Needless to say, such conclusions are hardly acceptable. Yet, logical deduction preserves truth but not necessarily intuitiveness, so let us take (4.16) and (4.17) for granted. Observe that from (4.17), (4.15), (4.16), and TBJ one immediately gets:

$$\mathsf{BJ}p \to \mathsf{K}p \tag{4.18}$$

and this, by (4.11), gives:

$$\mathsf{K}p \equiv \mathsf{BJ}p \tag{4.19}$$

According to formula (4.19), the tripartite account of propositional knowledge, TBJ, reduces to believing in being justified in believing. At first sight, it may look even appealing. However, (4.19) is not a "new" definition of knowledge, but characterizes a property of the standard tripartite concept of knowledge it exhibits assuming that formula (4.15) holds w.r.t. the components of the TBJ account. The Gettier counterexamples are still in place, not to mention the issues described in section 4.4 of this chapter.

4.5.5 The Scope Issue

The formula:

$(\mathbf{B}^*_{\mathsf{K}})$ $\quad \neg p \to \mathsf{K}\neg\mathsf{K}p$

plays a key role in the derivation of PDA presented above. As we remarked in section 4.5.1, in the current setting formula $\mathbf{B}^*_{\mathsf{K}}$ is proof-theoretically equivalent to formula $\mathbf{B}_{\mathsf{K}}$ as well as to formula $\mathbf{B}'_{\mathsf{K}}$, which, in turn, are epistemic versions of the Brouwerian formula/axiom **B**. Thus, taking into account the derivation of PDA presented above, we may say that the second problem arises when the formula $\mathtt{TB}^{+x}$ is the starting point, while the underlying epistemic-doxastic logic contains at least the logic $(\mathbf{KB})_{\mathsf{K}}$ in its purely epistemic component, at least $(\mathbf{KD})_{\mathsf{B}}$ in the doxastic component, and the linking formulas KB1, KB2 and BK2 as theorems/axioms.

[18] When the reading of $\mathsf{J}p$ is "p is justified," we get: (a) believing that p is justified yields that p is justified, and (b) believing that p is justified implies that p is the case.

Although the presence of formula $\mathbf{B}_\mathsf{K}$ in the epistemic component is sufficient here, the negative introspection formula $\mathbf{5}_\mathsf{K}$ will do as well.[19]

Interestingly, in both cases the factivity formula $\mathbf{T}_\mathsf{K}$ plays no role.[20]

4.6 The Third Problem

4.6.1 The Background

The appearance of the second problem described above heavily depends on the strength of the underlying epistemic logic in question: if an epistemic version of the Brouwerian formula/axiom **B** or the epistemic version of formula **5** are theorems/axioms of the logic, then, assuming that the epistemic-doxastic logic we operate with satisfies the remaining conditions specified above, the second problem shows up.

However, formulas $\mathbf{5}_\mathsf{K}$ and $\mathbf{B}_\mathsf{K}$ are almost unanimously rejected by epistemologists. The reasons are diverse. There is no space (and need) for presenting them in detail. Since the criticism of $\mathbf{5}_\mathsf{K}$ as a principle which applies to human cognitive agents is better known, let me only mention two arguments against $\mathbf{B}_\mathsf{K}$. The first, whose idea dates back to [21], is best summarized in [51] as follows:

> "... the agent should not be 'allowed' to create knowledge out of thin air. ([51], p. 121)"

One can argue, however, that under the notional reading of its respective parts formula $\mathbf{B}_\mathsf{K}$ is tenable (see footnote 7 above). The second argument against the formula in question, due to [54], is based on the observation

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19 As it is well-known (cf., e.g., [16], p. 114; the observation is due to [53]), formula $\mathbf{5}_\mathsf{K}$ together with formulas $\mathbf{D}_\mathsf{B}$ and KB1 yield the Paradox of The Perfect Believer, that is, the formula:

$$\mathsf{BK}p \rightarrow \mathsf{K}p \tag{4.20}$$

Now take a look at the derivation of PDA already presented above. Delete from it lines from (b.22) to (b.33), put instead a derivation of (4.20) based on formulas $\mathbf{5}_\mathsf{K}$, $\mathbf{D}_\mathsf{B}$ and KB1, and then continue as in the "old" derivation. The result is a derivation of PDA relying on the same premises as the "old" one, with the exception of formula $\mathbf{B}^*_\mathsf{K}$, now replaced with the negative introspection formula $\mathbf{5}_\mathsf{K}$ (although the linking formula KB2 now disappears, it is needed in order to obtain $\mathbf{4}_\mathsf{B}$).

20 Of course, in the current setting $\mathbf{5}_\mathsf{K}$ together with $\mathbf{T}_\mathsf{K}$ yields $\mathbf{B}^*_\mathsf{K}$. Thus one can derive the latter from $\mathbf{5}_\mathsf{K}$ and $\mathbf{T}_\mathsf{K}$, and then continue as in the case of the first derivation of PDA presented above, obtaining an unnecessarily tedious derivation of PDA.

that $\mathbf{B}_\mathsf{K}$ induces symmetry as the property of epistemic alternativeness in Kripke-style (epistemic) models, which is both unintuitive and leads to paradoxical consequences. But it cannot be said that **B** is always ignored in logico-philosophical considerations. Floridi's "logic of being informed" (cf. [13]) provides a notable example here.

The attitudes of logicians towards $\mathbf{5}_\mathsf{K}$ and $\mathbf{B}_\mathsf{K}$ are more nuanced. Both formulas are theorems of (epistemic) **S5**. Despite all known conceptual deficiencies of **S5** interpreted epistemically, the formalism of **S5** is still very often applied in logical considerations concerning knowledge, in particular knowledge dynamics and/or collective knowledge (cf. [52]). On the other hand, the remarkable result of [50], according to which: (a) any epistemic-doxastic logic which has formulas $\mathbf{D}_\mathsf{B}$, $\mathbf{5}_\mathsf{K}$, KB1, and KB3 as theorems/axioms[21], has the formula $\mathsf{B}p \equiv \mathsf{K}p$ as a theorem (that is, does not distinguish between belief and knowledge), but (b) for each proper subset of $\{\mathbf{D}_\mathsf{B}, \mathbf{5}_\mathsf{K}, \mathsf{KB1}, \mathsf{KB3}\}$, models can be build which invalidate $\mathsf{B}p \equiv \mathsf{K}p$, shows that having $(\mathbf{S5})_\mathsf{K}$ as a component of an epistemic-doxastic logic can be, to put it mildly, risky. When we add to this arguments against $\mathbf{B}_\mathsf{K}$, it becomes natural to consider, epistemically interpreted, modal logics which "lie between" **S4** and **S5** as candidates for epistemic components of an epistemic-doxastic logic. As Wolfgang Lenzen, much earlier, put it:

> "... the logic of knowledge must be at least as strong as system **S4.2** (...). Furthermore, the logic of knowledge must be at most as strong as system **S4.4** ..." ([27], p. 82)
>
> "... 'the' logic of knowledge most probably is **S4.2**." ([27], p. 83)

Recall that the logic of knowledge **S4.2**, that is, $(\mathbf{S4.2})_\mathsf{K}$, is the result of adding to $(\mathbf{S4})_\mathsf{K}$ the formula:

$(\mathbf{4.2}_\mathsf{K})$ $\quad \neg\mathsf{K}\neg\mathsf{K}p \to \mathsf{K}\neg\mathsf{K}\neg p$

as a new axiom.[22] Logic $(\mathbf{S4.4})_\mathsf{K}$, in turn, can be characterized as the extension of $(\mathbf{S4})_\mathsf{K}$ by the axiom:

$(\mathbf{4.4}_\mathsf{K})$ $\quad p \to (\neg\mathsf{K}p \to \mathsf{K}\neg\mathsf{K}p)$

[21] Formula KB3 is of the form: $\mathsf{B}p \to \mathsf{BK}p$.

[22] For the history, different formalizations, semantics, and other applications of the system **S4.2** see. e.g., [8].

Since neither $\mathbf{5}_{\mathsf{K}}$ nor $\mathbf{B}^{*}_{\mathsf{K}}$ are theorems of these logic as well as logics which "lie between" them, we cannot say that the second problem (presented in section 4.5) arises when one takes any of such logics as the epistemic component. This is the good news. But there is also a bad one: another problem shows up. This is due to some results of [19], [28], and [1]. Generally speaking, these results show that operating with $(\mathbf{S4.4})_{\mathsf{K}}$ or a weaker epistemic logic as the underlying logic of the knowledge operator may lead to the emergence of a variant of the "knowledge as true belief" account of propositional knowledge. By and large, this variant conceptualizes propositional knowledge as true conviction.

In order to present the relevant results in a concise, yet exact way, let me introduce some auxiliary notions first (these notions are not taken from the papers referred to above).

A wff A of the language of a propositional epistemic-doxastic logic is:

- a *κ-formula* iff the belief operator B does not occur in A,
- a *η-formula* iff the knowledge operator K does not occur in A,
- a μ-formula iff both the knowledge operator K and the belief operator B occur in A.

Note that the formulas in which neither K nor B occur, are both κ-formulas and η-formulas. This is intended.

We assume that an epistemic-doxastic logic in question has *interaction axioms*, which, syntactically, are μ-formulas. We do not assume, however, that each μ-formula being a theorem of L is an interaction axiom.

The *epistemic component* of a propositional epistemic-doxastic logic L comprises all the theorems of L which are κ-formulas. The *doxastic component* of the logic, in turn, consists of all the η-formulas which are its theorems.

Let L be an epistemic-doxastic logic based on CPL (henceforth: epistemic-doxastic logic). Let D be a formula of the language of L having the form $\mathsf{K}p \equiv A$, where K does not occur in A and B occurs in A. We say that the knowledge modality K is *reducible in L* to the belief modality B *by formula D* iff: (a) the set of theorems of L is included in the set of formulas which are L-derivable from the doxastic component of L together with the interaction axioms of L as well as the formula D, and (b) the set of η-formulas which are L-derivable from the doxastic component of L together with the interaction axioms of L as well as the formula D is included in the doxastic component of L. (This concept is borrowed from [19], although its definition has been slightly reformulated here.) Reducibility so conceived is a kind of definability in a multimodal setting.

The result of [19] which is relevant for the purposes of this chapter, can now be worded as follows:

> $(*)$ if L is an epistemic-doxastic logic closed under the rules of necessitation N_{B} and N_{K} such that: (i) the doxastic component of L is the logic $(\mathbf{KD45})_{\mathsf{B}}$, (ii) the epistemic component of L is included in the logic $(\mathbf{S4.4})_{\mathsf{K}}$ and (iii) the linking formulas KB1 and KB2 are theorems of L, then the knowledge modality K is reducible in L to the belief modality by the formula
>
> $$\mathsf{K}p \equiv p \wedge \mathsf{B}p \tag{4.21}$$

For the original formulation, cf. [19], Theorem 4.2. As a matter of fact, $(*)$ expresses only the first claim of the theorem. The theorem also states that one gets reducibility of the above kind if the epistemic component is isomorphic to an alethic modal propositional logic that is not stronger than **S4.4**.

As $(\mathbf{S4.2})_{\mathsf{K}}$ is included in $(\mathbf{S4.4})_{\mathsf{K}}$, the result $(*)$ pertains also to the case in which the epistemic component is just $(\mathbf{S4.2})_{\mathsf{K}}$, "the" logic of knowledge according to Lenzen's claim.

Aucher (cf. [1], Theorem 5.6.4), in turn, observed that:

> $(**)$ if L is an epistemic-doxastic logic closed under the rules of necessitation N_{B} and N_{K} such that: (i) the doxastic component of L is the logic $(\mathbf{KD45})_{\mathsf{B}}$, (ii) the epistemic component of L is the logic $(\mathbf{S4.4})_{\mathsf{K}}$, and (iii) the linking formulas KB1 and KB2 are theorems of L, and (iv) the following linking formula:
>
> (KB3) $\mathsf{B}p \rightarrow \mathsf{BK}p$
>
> is a theorem of L, then the knowledge modality K is explicitly defined in L in terms of the belief modality B by means of the formula (4.21), which is a theorem of L.[23]

[23] Again, this is only the first part of the theorem (whose proof Aucher credits

In both cases, (∗) and (∗∗), the doxastic component of L is the logic $(\mathbf{KD45})_{\mathsf{B}}$. This logic is commonly regarded as the logic of a special kind of belief, namely strong/firm belief or *conviction*. Taking this into account, we may say that assumptions concerning the underlying epistemic-doxastic logic yield that the "knowledge as true conviction" account of propositional knowledge comes into play, directly (due to (∗∗)) or indirectly (because of (∗)). One may wonder whether such an account is a variant of the "true belief plus something else" account. Yet, regardless of the answer, the former raises some new issue.

4.6.2 The Third Problem Itself

The third problem I am going to point out in this chapter, is:

(III) *The "knowledge as true conviction" account of propositional knowledge disqualifies each conviction concerning an epistemically unresolved matter.*

We face this problem due to the appearance of the Doxastic Misfortune Paradox characterized below.

4.6.3 The Doxastic Misfortune Paradox

For clarity, let us use the letter C for the conviction operator. The language of analysis differs from that described in section 4.3 in the presence

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to Lenzen). The second part speaks about the lack of explicit definability of this kind is some weaker epistemic-doxastic logics (cf. [1], p. 121). By the way, as for the theoremhood of formula (4.21), the linking formula KB2 is superfluous. The formula:

$$p \to (\neg\mathsf{B}\neg\mathsf{B}p \equiv \mathsf{K}p) \tag{4.22}$$

is derivable in any epistemic-doxastic logic closed under the rule N_{B} which has $(\mathbf{KD45})_{\mathsf{B}}$ as the doxastic component, $(\mathbf{S4.4})_{\mathsf{K}}$ as the epistemic component, and the linking formulas KB1 and KB3 as theorems (cf. [63], p. 313-314). As for $(\mathbf{KD45})_{\mathsf{B}}$, formulas $\neg\mathsf{B}\neg\mathsf{B}p$ and $\mathsf{B}p$ are equivalent. Thus we get:

$$p \to (\mathsf{B}p \equiv \mathsf{K}p) \tag{4.23}$$

and hence

$$p \wedge \mathsf{B}p \to \mathsf{K}p \tag{4.24}$$

On the other hand, (4.23) together with $\mathbf{T}_{\mathsf{K}}$ gives:

$$\mathsf{K}p \to p \wedge \mathsf{B}p \tag{4.25}$$

Having (4.23) and (4.25), we also have (4.21).

of C instead of B, and the disappearance of Ω. The remaining assumptions concerning logical preliminaries hold accordingly. The formula:

(TC) $\quad \mathsf{K}p \equiv p \wedge \mathsf{C}p$

expresses the "knowledge as true conviction" account of knowledge that.

Now observe that the following formulas:

$$\neg\mathsf{K}\neg p \wedge \mathsf{C}\neg p \to p \tag{4.26}$$

$$\neg\mathsf{K}p \wedge \mathsf{C}p \to \neg p \tag{4.27}$$

are derivable from the formula TC. Here is a derivation of formula (4.26):[24]

c.1.	$\mathsf{K}p \equiv p \wedge \mathsf{C}p$	(TC)
c.2.	$p \wedge \mathsf{C}p \to \mathsf{K}p$	(c.1. CPL)
c.3.	$p \to (\mathsf{C}p \to \mathsf{K}p)$	(c.2. CPL)
c.4.	$\neg p \to (\mathsf{C}\neg p \to \mathsf{K}\neg p)$	(c.3. $p/\neg p$)
c.5.	$\neg(\mathsf{C}\neg p \to \mathsf{K}\neg p) \to p$	(c.4. CPL)
c.6.	$\mathsf{C}\neg p \wedge \neg\mathsf{K}\neg p \to p$	(c.5. CPL)
(4.26)	$\neg\mathsf{K}\neg p \wedge \mathsf{C}\neg p \to p$	(c.6. CPL)

The derivation of formula (4.27) proceeds analogously, viz.:

c.7.	$\mathsf{K}p \equiv p \wedge \mathsf{C}p$	(TC)
c.8.	$p \wedge \mathsf{C}p \to \mathsf{K}p$	(c.7. CPL)
c.9.	$p \to (\mathsf{C}p \to \mathsf{K}p)$	(c.8. CPL)
c.10.	$p \to (\neg\mathsf{K}p \to \neg\mathsf{C}p)$	(c.9. CPL)
c.11.	$\neg(\neg\mathsf{K}p \to \neg\mathsf{C}p) \to \neg p$	(c.10. CPL)
(4.27)	$\neg\mathsf{K}p \wedge \mathsf{C}p \to \neg p$	(c.11. CPL)

Note that in both cases no specific assumption concerning the behaviour of the conviction operator C is used. Only CPL-means are employed.

Let us now introduce the concept of *knowing whether*, in symbols K^{w}. This will be done in the standard way, by means of the equivalence:

$$\mathsf{K}^{\mathsf{w}}p \equiv \mathsf{K}p \vee \mathsf{K}\neg p \tag{4.28}$$

[24] This derivation is very much alike the derivation of formula PAC given in section 4.4, but in order to make this section self-contained we present it here anyway.

The expression $\neg \mathsf{K}^{\mathsf{w}} p$ can thus be read informally as "p is epistemically unresolved." Clearly, we have:

$$\neg \mathsf{K}^{\mathsf{w}} p \rightarrow \neg \mathsf{K} p \tag{4.29}$$

and

$$\neg \mathsf{K}^{\mathsf{w}} p \rightarrow \neg \mathsf{K} \neg p \tag{4.30}$$

Given all this, the following formula becomes derivable:

(DMP) $\quad \neg \mathsf{K}^{\mathsf{w}} p \rightarrow (\mathsf{C}\neg p \rightarrow p) \wedge (\mathsf{C}p \rightarrow \neg p)$

Here is a (exemplary) derivation of DMP:

c.12.	$\neg \mathsf{K}\neg p \wedge \mathsf{C}\neg p \rightarrow p$	(4.26)
c.13.	$\neg \mathsf{K}\neg p \rightarrow (\mathsf{C}\neg p \rightarrow p)$	(c.12. CPL)
c.14.	$\neg \mathsf{K}^{\mathsf{w}} p \rightarrow \neg \mathsf{K}\neg p$	(4.30)
c.15.	$\neg \mathsf{K}^{\mathsf{w}} p \rightarrow (\mathsf{C}\neg p \rightarrow p)$	(c.14., c.13. CPL)
c.16.	$\neg \mathsf{K}p \wedge \mathsf{C}p \rightarrow \neg p$	(4.27)
c.17.	$\neg \mathsf{K}p \rightarrow (\mathsf{C}p \rightarrow \neg p)$	(c.16. CPL)
c.18.	$\neg \mathsf{K}^{\mathsf{w}} p \rightarrow \neg \mathsf{K}p$	(4.29)
c.19.	$\neg \mathsf{K}^{\mathsf{w}} p \rightarrow (\mathsf{C}p \rightarrow \neg p)$	(c.18., c.17. CPL)
(DMP)	$\neg \mathsf{K}^{\mathsf{w}} p \rightarrow (\mathsf{C}\neg p \rightarrow p) \wedge (\mathsf{C}p \rightarrow \neg p)$	(c.15., c.19 CPL)

According to formula DMP, if it is not known whether p (i.e. neither p nor $\neg p$ is known), then conviction that $\neg p$, if held, is wrong, and conviction that p, if held, is wrong. In other words, if it is not known whether p, then p is the case if $\neg p$ is strongly believed, and $\neg p$ is the case if p is strongly believed. To speak generally, when p remains epistemically unresolved, each conviction about its logical value, if held, is wrong. Formula DMP is paradoxical in its disallowance for having true convictions about a subject matter which is not epistemically resolved. It seems appropriate to coin it the *Doxastic Misfortune Paradox*, which explains the acronym DMP.

As the derivation of DMP presented above illustrates, one can get it by CPL-means only, without relying on any specific assumptions concerning the logic that governs the conviction operator.

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4.6.4 Consistency of the Antecedents

Assuming that derivability in L satisfies the condition (♣) (see section 4.4.2) and is closed under the rule $E_{\neg\neg}$ of elimination of double negations, the antecedents of formulas (4.26) and (4.27) are L-consistent if L differentiates between conviction and knowledge, that is, formula $\mathsf{C}p \to \mathsf{K}p$ is not a theorem of L.

Concerning the issue of consistency of the antecedent of DMP, the antecedent is L-consistent if the formula $\neg\mathsf{K}\neg p \to \mathsf{K}p$ is L-refuted. But the formula is not a theorem of $(\mathbf{S5})_{\mathsf{K}}$, and hence of epistemic logics included in it.

4.7 Summary and Conclusion

Let us summarize. In the case of a tripartite (standard or not) account of propositional knowledge, the following problems were diagnosed:

(I) *For an epistemically possible proposition to hold, it suffices that its classical negation is believed and satisfies the third clause of the definition of knowledge, whatever this clause occurs to be.*

(II) *Knowing that a proposition holds reduces to believing that it holds together with believing that the proposition satisfies the third clause of the definition of knowledge, whatever this clause occurs to be.*

Problem (I) arises due to the Paradox of Astounding Consequent (recall that by Ω one expresses the third, "missing" clause of a tripartite definition of propositional knowledge; K and B stand for knowledge operator and the belief operator, respectively):

(PAC) $\neg\mathsf{K}\neg p \wedge (\mathsf{B}\neg p \wedge \Omega\neg p) \to p$

One can get PAC from the formula expressing the generalized tripartite account:

($\mathtt{TB}^{+x}$) $\mathsf{K}p \equiv (p \wedge \mathsf{B}p) \wedge \Omega p$

by CPL-means only: no specific assumptions concerning K, B and Ω are needed.

Problem (II) stems from what I have called the Paradox of Doxastic Agency:

(PDA) $\mathsf{K}p \equiv \mathsf{B}p \wedge \mathsf{B}\Omega p$

However, PDA is not just a CPL-consequence of $\mathtt{TB}^{+x}$, but relies on some specific assumptions concerning knowledge, belief, and their interactions. If you do not share these assumptions, problem (II) does not bother you.

As for the "knowledge as true conviction" account, the following problem was diagnosed:

(III) *The "knowledge as true conviction" account of propositional knowledge disqualifies each conviction concerning an epistemically unresolved matter.*

Problem (III) arises due to what I have dubbed the Doxastic Misfortune Paradox (recall that K^{w} refers to knowledge-whether, while C stands for the conviction operator):

(DMP) $\neg\mathsf{K}^{\mathsf{w}}p \rightarrow (\mathsf{C}\neg p \rightarrow p) \wedge (\mathsf{C}p \rightarrow \neg p)$

DMP does not rely on any specific assumptions concerning the conviction operator: one gets it by CPL-means if only knowledge is defined as true conviction and knowing-whether is conceived in the standard way.

This is the diagnosis. But a diagnosis should be accompanied with recommendations. My recommendation is a modest one: beware the problems pointed out in this chapter when aiming at a satisfactory analysis of propositional knowledge.

Having this in mind, in the next chapter I propose a non-standard account of epistemic logic broadly conceived.

Chapter 5

Being Epistemically Permitted

5.1 Introduction

We are often confronted with a number of alternative accounts of how things are, yet without knowing which of the accounts, if any, is the right one. These accounts disagree on some issues and agree on others. Despite discrepancies, however, some facts still remain known, some states of affairs are considered impossible, and some statements are *epistemically permitted* while other are not.

In this chapter I define the relation "a declarative sentence is epistemically permitted by a set of possible worlds" and I characterize its basic properties. The possible worlds in question are supposed to represent alternative accounts of how things are. For this reason I dub the relation "epistemic permittance" or "e-permittance" for short. The definition proposed is an explication of the corresponding intuitive notion of being epistemically permitted, taken in one of its meanings. Basic intuitions are presented in section 5.1.2 below.

The concept of epistemic permittance enables an introduction, as a by-product, of some concept of *knowledge* free of the drawbacks pointed out in the previous chapter. However, this is only one of possible gains one gets from an account of epistemic permittance.

In order to make this chapter self-contained, let me start with a short description of the basic logical tools used.

5.1.1 Logical Preliminaries

We remain at the propositional level only. We consider a non-modal propositional language, $\mathcal{L}$. The vocabulary of $\mathcal{L}$ includes a non-empty set $\mathcal{P} = \{p, q, r, \ldots\}$ of propositional variables, the propositional constant $\perp$ (*falsum*), and the connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$. Well-formed formulas (wffs for short) of $\mathcal{L}$ are defined in the usual manner. We shall use the letters A, B, C, D, ..., with subscripts if needed, as metalanguage variables for wffs of $\mathcal{L}$. The letters X, Y, ... are metalanguage variables for sets of wffs of $\mathcal{L}$.

The connectives, as well as $\perp$, are understood, at the truth-functional level, as in Classical Propositional Logic. By an $\mathcal{L}$-*model* we mean an ordered pair $M = \langle \mathcal{W}, \mathcal{V} \rangle$, where $\mathcal{W} \neq \emptyset$ and $\mathcal{V} : \mathcal{P} \times \mathcal{W} \longmapsto \{\mathbf{1}, \mathbf{0}\}$ is a valuation of propositional variables in $\mathcal{P}$ w.r.t. elements of $\mathcal{W}$. As usual, $\mathcal{W}$ is called the domain of an $\mathcal{L}$-model $M = \langle \mathcal{W}, \mathcal{V} \rangle$, and the elements of $\mathcal{W}$ are called possible worlds of the model. We put:

- $\mathcal{V}(\perp, w) = \mathbf{0}$ for each $w \in \mathcal{W}$.

The concept of truth of a wff A in a world $w \in \mathcal{W}$ of M, in symbols $M, w \models A$, is defined in the standard manner. The letter p stands below for a metalinguistic variable for propositional variables.

Definition 5.1 (Truth of a wff in a world).

1. $M, w \models \perp$ *iff* $\mathcal{V}(\perp, w) = \mathbf{1}$,
2. $M, w \models \mathsf{p}$ *iff* $\mathcal{V}(\mathsf{p}, w) = \mathbf{1}$,
3. $M, w \models \neg B$ *iff it is not the case that* $M, w \models B$,
4. $M, w \models (B \wedge C)$ *iff* $M, w \models B$ *and* $M, w \models C$,
5. $M, w \models (B \vee C)$ *iff* $M, w \models B$ *or* $M, w \models C$,
6. $M, w \models (B \rightarrow C)$ *iff* $M, w \models \neg B$ *or* $M, w \models C$,
7. $M, w \models (B \equiv C)$ *iff* $M, w \models B \rightarrow C$ *and* $M, w \models C \rightarrow B$.

The inscription $M \models A$ means "A is true in an $\mathcal{L}$-model $M = \langle \mathcal{W}, \mathcal{V} \rangle$."

Definition 5.2 (Truth in an $\mathcal{L}$-model). *$M \models A$ iff $M, w \models A$ for each $w \in W$.*

Elements of domains of $\mathcal{L}$-models, the possible worlds, will be intuitively thought of here as *alternative accounts of how things are.* Their alternativeness amounts to the effect that any two distinct possible worlds:

w, w' in the domain of an $\mathcal{L}$-model disagree on at least one wff, that is, the wff has distinct logical values in w and in w'. This has no impact on the formalism, however. As long as we remain at the propositional level, the only condition imposed on $\mathcal{W}$ is non-emptiness. It follows that the domain of an $\mathcal{L}$-model need not contain all the relevant alternatives.

By a *state* we will mean a non-empty set of possible worlds. In view of the intuitive interpretation of possible worlds adopted above, a non-singleton state comprises a number of alternative accounts of how things are.

Let $M = \langle \mathcal{W}, \mathcal{V} \rangle$ be an $\mathcal{L}$-model.

Definition 5.3 (Truth set of a wff in an $\mathcal{L}$-model)**.**

$$|A|_M = \{w \in \mathcal{W} : M, w \models A\}$$

Of course, $|\bot|_M = \emptyset$.

Definition 5.4 (M-state)**.** *An M-state is a non-empty subset of $\mathcal{W}$.*

Note that $\mathcal{W}$ is (also) an M-state, and that, for each $w \in \mathcal{W}$, the singleton set $\{w\}$ is an M-state.

5.1.2 Intuitions

Our basic intuition concerning the analysed concept of being epistemically permitted is:

(I) *A declarative sentence/wff $\mathcal{D}$ is epistemically permitted by state $\mathcal{S}$ iff it is not the case that $\mathcal{S}$ rules out $\mathcal{D}$.*

However, what "rules out" means depends on the form of $\mathcal{D}$.

$\mathcal{D}$ can be *positive*, that is not of the form $\neg\mathcal{E}$ (where $\neg$ stands for sentential negation and $\mathcal{E}$ is a declarative sentence/wff). It is natural to postulate:

(II) *Let $\mathcal{D}$ be positive. State $\mathcal{S}$ rules out $\mathcal{D}$ iff $\mathcal{D}$ is false in each world of $\mathcal{S}$.*

For example, "Andrew is a bachelor" is ruled out by a state which comprises (only) possible worlds in which Andrew is married.

$\mathcal{D}$ can be *negative* that is of the form $\neg\mathcal{E}$, where $\mathcal{E}$ is positive.[25] We seem justified in saying:

[25] Observe that $\neg\neg\mathcal{E}$ is neither negative nor positive. We will come back to this issue later on.

(III) *Let $\mathcal{D}$ be negative and $\mathcal{D} = \neg\mathcal{E}$. State $\mathcal{S}$ rules out $\mathcal{D}$ iff $\mathcal{E}$ is true in some world of $\mathcal{S}$.*

For instance, a state that contains a possible world in which Andrew is a bachelor rules out the sentence "It is not the case that Andrew is a bachelor."

Assuming bivalence, by **(I)** and **(II)** we get:

(II*) *A positive, $\mathcal{D}$, is epistemically permitted by state $\mathcal{S}$ iff $\mathcal{D}$ is true in some world of $\mathcal{S}$.*

By **(I)** and **(III)**, in turn, we get:

(III*) *A negative, $\mathcal{D}$, is epistemically permitted by state $\mathcal{S}$ iff $\mathcal{D}$ is true in each world of $\mathcal{S}$.*

An analogy may be of help. A civil servant is permitted to issue a positive decision if there is a rule that entitles him/her to do so, and is permitted to decide to the negative if the disputed activity is forbidden by each rule that is applicable to the case. Similarly, a negative is epistemically permitted by a state if there is no world of the state that makes the negated sentence true, while for a positive being epistemically permitted by a state amounts to the existence of a world of the state which makes it true. Our usage of "being permitted" is thus akin to that of its deontic cousin. Yet, we do not aim at analysing "being permitted" deontically construed. Epistemic permittance is a relation between a declarative sentence/wff on the one hand, and a state on the other. What is (or is not) epistemically permitted is a declarative sentence/wff, and what permits it (or does not permit) is a set of possible worlds, where possible worlds are intuitively thought of as alternative accounts of how things are.[26]

5.2 Epistemic Permittance

5.2.1 Definition and Basic Properties

We use $\looparrowright$ as the sign of epistemic permittance relation. Given the considerations presented above, we define the relation as follows:

[26] Looking from a formal point of view, epistemic permittance belongs to the same category as *support* analysed in Inquisitive Semantics (see, e.g., [11]). However, the underlying intuitions are different. Moreover, Inquisitive Semantics conceives states/sets of possible worlds as information states.

Definition 5.5 (Epistemic permittance; e-permittance). *Let* $M = \langle \mathcal{W}, \mathcal{V} \rangle$ *be an* $\mathcal{L}$*-model, and let* $\mathcal{S}$ *be an* M*-state.*

1. $\mathcal{S} \looparrowright \mathtt{p}$ *iff* $|\mathtt{p}|_M \cap \mathcal{S} \neq \emptyset$*, for any propositional variable* $\mathtt{p}$*;*
2. $\mathcal{S} \looparrowright \neg A$ *iff* $\mathcal{S} \not\looparrowright A$*;*
3. $\mathcal{S} \looparrowright (A \vee B)$ *iff* $|(A \vee B)|_M \cap \mathcal{S} \neq \emptyset$*;*
4. $\mathcal{S} \looparrowright (A \wedge B)$ *iff* $|(A \wedge B)|_M \cap \mathcal{S} \neq \emptyset$*;*
5. $\mathcal{S} \looparrowright (A \to B)$ *iff* $|(A \to B)|_M \cap \mathcal{S} \neq \emptyset$*;*
6. $\mathcal{S} \looparrowright \bot$ *iff* $|\bot|_M \cap \mathcal{S} \neq \emptyset$*.*

Observe that epistemic permittance *is not* defined inductively. This is intended.

For positive wffs, being permitted by a state amounts to being true in some world(s) of the state. To be more precise, as an immediate consequence of Definition 5.5 we get:

Corollary 5.1. *Let* $\mathcal{S}$ *be an* M*-state and let* A *be a positive wff. Then* $\mathcal{S} \looparrowright A$ *iff* $M, w \models A$ *for some* $w \in \mathcal{S}$*.*

However, the case of negative wffs is different. By Corollary 5.1 and clause (*2*) of Definition 5.5 we have:

Corollary 5.2. *Let* $\mathcal{S}$ *be an* M*-state. Let* D *be a wff of any of the forms:* $\mathtt{p}, \bot, (B \vee C), (B \wedge C), (B \to C)$*. Then* $\mathcal{S} \looparrowright \neg D$ *iff* $M, w \not\models D$ *for each* $w \in \mathcal{S}$*.*

Hence:

Corollary 5.3. *Let* $\mathcal{S}$ *be an* M*-state and let* A *be a negative wff. Then* $\mathcal{S} \looparrowright A$ *iff* $M, w \models A$ *for each* $w \in \mathcal{S}$*.*

Corollary 5.3 shows that negatives behave in the context of e-permittance as it has been required in section 5.1.2.

But what about wffs which are neither positive nor negative? As for $\mathcal{L}$, there is only one kind of such wffs, namely wffs falling under the general schema:

$$\neg \ldots \neg D \tag{5.1}$$

where D is positive and the number of negations preceding D is greater than 1. If the number is even, we say that (5.1) is a $\neg_e$-wff; otherwise (5.1) is a $\neg_o$-wff. By D_A we designate the positive wff which occurs in a $\neg_e$-wff A or in a $\neg_o$-wff A after the string of negations.[27]

[27] When A is neither positive nor negative, D_A is in the scope of the rightmost negation of the string.

One can prove:

Corollary 5.4. *For each $\mathcal{L}$-model M and each M-state $\mathcal{S}$:*

$$\mathcal{S} \looparrowright \neg\neg A \text{ iff } \mathcal{S} \looparrowright A$$

Proof. By the clause (2) of Definition 5.5 we have:

$$\mathcal{S} \looparrowright \neg\neg A \text{ iff } \mathcal{S} \not\looparrowright \neg A$$

$$\mathcal{S} \not\looparrowright \neg A \text{ iff } \mathcal{S} \looparrowright A$$

and hence $\mathcal{S} \looparrowright \neg\neg A$ iff $\mathcal{S} \looparrowright A$. □

Thus, taking into account corollaries 5.1, 5.2, and 5.4 we get:

Corollary 5.5.

1. *Let A be a $\neg_e$-wff. Then $\mathcal{S} \looparrowright A$ iff $M, w \models D_A$ for some $w \in \mathcal{S}$ iff $M, w \models A$ for some $w \in \mathcal{S}$.*
2. *Let A be a $\neg_o$-wff. Then $\mathcal{S} \looparrowright A$ iff $M, w \not\models D_A$ for each $w \in \mathcal{S}$ iff $M, w \models A$ for each $w \in \mathcal{S}$.*

For brevity, let us introduce:

Definition 5.6 (p-wffs and n-wffs)**.**

1. *A p-wff is a positive wff or a $\neg_e$-wff.*
2. *A n-wff is a negative wff or a $\neg_o$-wff.*

As we have shown, the categories of p-wffs and n-wffs are semantically homogeneous with regard to e-permittance. A p-wff is e-permitted by a state iff it is true in at least one world of the state, while a n-wff is e-permitted by a state iff it is true in each world of the state. E-permittance could had been concisely defined in terms of p-wffs and n-wffs. However, doing this would require an *ad hoc* acceptance of the claim of Corollary 5.4.

As an immediate consequence of the above corollaries we get:

Corollary 5.6. *Let $M = \langle \mathcal{W}, \mathcal{V} \rangle$ be an $\mathcal{L}$-model and $\{w\}$ be a (singleton) M-state. Then $\{w\} \looparrowright A$ iff $M, w \models A$.*

Remark 5.1. For a singleton state e-permittance amounts to truth in the only world of the state.

Remark 5.2. Epistemic permittance becomes intensional when non-singleton states enter the picture. It happens that wffs which have equal truth sets (i.e. are classically equivalent) are not simultaneously permitted by a state. For example, we have: $|\neg(p \rightarrow q)|_M = |p \wedge \neg q|_M$. Now take an $\mathcal{L}$-model and its state $\{w_1, w_2\}$ such that:

- $\mathcal{V}(p, w_1) = \mathbf{1}$ and $\mathcal{V}(q, w_1) = \mathbf{0}$,
- $\mathcal{V}(p, w_2) = \mathbf{0}$ and $\mathcal{V}(q, w_2) = \mathbf{0}$.

We get:

$$\{w_1, w_2\} \not\looparrowright \neg(p \rightarrow q)$$

$$\{w_1, w_2\} \looparrowright p \wedge \neg q$$

Remark 5.3. Note that wffs of the forms:

$$\neg A \tag{5.2}$$

$$A \rightarrow \bot \tag{5.3}$$

do no differ as to their truth conditions in a world, but can differ with respect to e-permittance by states. When A is a p-wff, (5.2) is permitted only by a state in which A is false in each world of the state, whereas (5.3) can be permitted by a state in which A is false only in some, but not all worlds. This does not mean, however, that the negation connective $\neg$ has a non-classical meaning in $\mathcal{L}$. Its meaning is determined by the standard truth condition. But $\neg$ behaves in a somewhat non-standard way in the context of e-permittance.

Note also that in general e-permittance is neither downward closed (if A is a p-wff, e-permittance of A by $\mathcal{S}$ need not yield e-permittance of A by a proper subset of $\mathcal{S}$) nor upward closed (a n-wff permitted by a state need not be permitted by an extension of the state). However, e-permittance is upward closed for p-wffs and downward closed in the case of n-wffs.

5.3 Modalization

Let us now augment the initial language $\mathcal{L}$ with the modalities $\Box$ (necessity) and $\Diamond$ (possibility). Wffs of the enriched language are defined in the standard manner. We label the new language as $\mathfrak{L}$. In this chapter we use $\phi, \psi, \ldots$ as metalanguage variables for wffs of $\mathfrak{L}$, and $\boldsymbol{\Phi}, \boldsymbol{\Psi}, \ldots$

as metalanguage variables for sets of wffs of the language. Whenever $\Box$ or $\Diamond$ precedes a metalanguage expression referring to wffs of $\mathcal{L}$, it is understood that the wff in the scope of a modality belongs to $\mathcal{L}$ (i.e. is a wff of $\mathfrak{L}$ in which no modality occurs).

Definition 5.7 (**S5**-model)**.** *An* **S5**-*model is a structure:*

$$\langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$$

where $\mathcal{W} \neq \emptyset$, $\mathcal{V}$ is a valuation of $\mathcal{P}$ w.r.t. elements of $\mathcal{W}$, and $\mathcal{R} = \mathcal{W} \times \mathcal{W}$, that is, $\mathcal{R}$ is universal in $\mathcal{W}$.

Thus by **S5**-models we will mean here only these relational models in which the accessibility relation $\mathcal{R}$ is universal. In the case of **S5**-models we have:[28]

$$\mathcal{M}, w \models \Box\phi \textit{ iff } \mathcal{M}, w \models \phi \textit{ for every } w \in \mathcal{W}, \tag{5.4}$$

$$\mathcal{M}, w \models \Diamond\phi \textit{ iff } \mathcal{M}, w \models \phi \textit{ for some } w \in \mathcal{W}. \tag{5.5}$$

The remaining truth conditions are alike these specified by Definition 5.1. A wff A is true in an **S5**-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ (in symbols: $\mathcal{M} \models A$) iff $\mathcal{M}, w \models A$ for every $w \in \mathcal{W}$.

It is well-known that the modal propositional logic **S5** is sound and complete w.r.t. the class of models of the above kind.

Definition 5.8 (Accompanied **S5**-model)**.** *Let $M = \langle \mathcal{W}, \mathcal{V} \rangle$ be an $\mathcal{L}$-model, $\mathcal{S}$ be an M-state, and $\mathcal{R} = \mathcal{W} \times \mathcal{W}$. Let $\mathcal{M}_{\mathcal{S}}$ be an* **S5**-*model such that:*

$$\mathcal{M}_{\mathcal{S}} = \langle \mathcal{S}, \mathcal{R}|\mathcal{S}, \mathcal{V}|\mathcal{S} \rangle$$

$\mathcal{M}_{\mathcal{S}}$ is called the **S5**-*model accompanied with the $\mathcal{L}$-model M w.r.t. the M-state $\mathcal{S}$.*

Clearly, the relation $\mathcal{R}|\mathcal{S}$ (that is, the restriction of the relation $\mathcal{R}$ to $\mathcal{S}$) is universal in $\mathcal{S}$, and the valuation $\mathcal{V}|\mathcal{S}$ of $\mathcal{M}_{\mathcal{S}}$ agrees with the valuation $\mathcal{V}$ of M on all worlds in $\mathcal{S}$.

It is obvious that for each $\mathcal{L}$-model M and each state of the model there exists exactly one **S5**-model accompanied with M w.r.t. the state. For each wff A of $\mathcal{L}$ we have:

Corollary 5.7. *Let M be an $\mathcal{L}$-model, $\mathcal{S}$ be an M-state, and $w \in \mathcal{S}$. Let A be a wff of $\mathcal{L}$. Then $M, w \models A$ iff $\mathcal{M}_{\mathcal{S}}, w \models A$.*

The following is true as well:

[28] As above, "$\mathcal{M}, w \models \psi$" means: "$\psi$ is true in world w of an **S5**-model $\mathcal{M}$."

Lemma 5.1. *For each M-state $\mathcal{S}$:*

1. *if A is a p-wff of $\mathcal{L}$, then: $\mathcal{S} \looparrowright A$ iff $\mathcal{M}_{\mathcal{S}} \models \Diamond A$,*
2. *if A is a n-wff of $\mathcal{L}$, then: $\mathcal{S} \looparrowright A$ iff $\mathcal{M}_{\mathcal{S}} \models \Box A$.*

Proof. As for (*1*), it suffices to recall that for a p-wff A we have $\mathcal{S} \looparrowright A$ iff A is true in at least one world of $\mathcal{S}$. By Corollary 5.5, the same holds for $\neg_e$-wffs. On the other hand, the accessibility relation in $\mathcal{M}_{\mathcal{S}}$ is universal in $\mathcal{S}$ and thus $\mathcal{M}_{\mathcal{S}} \models \Diamond A$ iff $\mathcal{M}_{\mathcal{S}}, w \models A$ for at least one $w \in \mathcal{S}$.

Clause (*2*) is an immediate consequence of Corollary 5.3, Corollary 5.5, and the fact that $\mathcal{R}|\mathcal{S}$ is universal in $\mathcal{S}$. □

Let us also prove:

Lemma 5.2. *Let A be a wff of $\mathcal{L}$. For each M-state $\mathcal{S}$:*

1. *$\mathcal{S} \looparrowright \neg(A \to \bot)$ iff $\mathcal{M}_{\mathcal{S}} \models \Box A$,*
2. *$\mathcal{S} \looparrowright (\neg A \to \bot)$ iff $\mathcal{M}_{\mathcal{S}} \models \Diamond A$.*

Proof. As for (*1*), $\neg(A \to \bot)$ is a n-wff and hence, by Corollary 5.3, $\mathcal{S} \looparrowright \neg(A \to \bot)$ iff for each $w \in \mathcal{S}$: $M, w \not\models (A \to \bot)$, that is, $\mathcal{M}_{\mathcal{S}}, w \models A$ for any $w \in \mathcal{S}$, which, due to the universality of $\mathcal{R}|\mathcal{S}$ in $\mathcal{S}$, gives $\mathcal{M}_{\mathcal{S}}, w \models \Box A$ for any $w \in \mathcal{S}$, that is, $\mathcal{M}_{\mathcal{S}} \models \Box A$.

Concerning (*2*): $\mathcal{S} \looparrowright (\neg A \to \bot)$ iff $|(\neg A \to \bot)|_M \cap \mathcal{S} \neq \emptyset$ iff for some $w \in \mathcal{S}$: $M, w \models A$ iff $\mathcal{M}_{\mathcal{S}}, w \models \Diamond A$ for some $w \in \mathcal{S}$. Due to the universality of $\mathcal{R}|\mathcal{S}$ in $\mathcal{S}$, the clauses: "$\mathcal{M}_{\mathcal{S}}, w \models \Diamond A$ for some $w \in \mathcal{S}$" and "$\mathcal{M}_{\mathcal{S}} \models \Diamond A$" are equivalent. □

5.4 Knowledge in a State

As it is well-known, **S5** can be interpreted as an epistemic logic, where the box, $\Box$, represents the knowledge operator, and the diamond, $\Diamond$, represents, generally speaking, epistemic possibility. This suggests a kind of purely epistemic readings of some *metalanguage* expressions of the form "$\mathcal{S} \looparrowright B$," where B is a wff of $\mathcal{L}$.

In this section we assume that M is an arbitrary but fixed $\mathcal{L}$-model, and that $\mathcal{S}$ is an M-state.

Suppose that B is of the form:

$$\neg(A \to \bot) \tag{5.6}$$

Consider:

$$\mathcal{S} \looparrowright \neg(A \to \bot) \tag{5.7}$$

Due to clause (1) of Lemma 5.2, this can be read:

A constitutes an item of knowledge in a state $\mathcal{S}$

where “A constitutes an item of knowledge in a state $\mathcal{S}$” means:

$$\mathcal{M}_{\mathcal{S}} \models \Box A \tag{5.8}$$

which, in turn, is equivalent with:

$$A \textit{ is true in every world of state } \mathcal{S}. \tag{5.9}$$

because $\mathcal{R}|\mathcal{S}$ is universal in $\mathcal{S}$.

Observe that, according to Lemma 5.2, (5.8) does not differentiate between n-wffs and p-wffs.

Now assume that B has the form:

$$\neg A \to \bot \tag{5.10}$$

Consider:

$$\mathcal{S} \looparrowright (\neg A \to \bot) \tag{5.11}$$

By Lemma 5.2, (5.11) amounts to:

$$\mathcal{M}_{\mathcal{S}} \models \Diamond A \tag{5.12}$$

which, again due to the universality of $\mathcal{R}|\mathcal{S}$ in $\mathcal{S}$, is equivalent with:

$$A \textit{ is true in some world of } \mathcal{S} \tag{5.13}$$

Thus it seems natural to read (5.11) as follows:

$$A \textit{ is epistemically possible in a state } \mathcal{S} \tag{5.14}$$

irrespective of whether A is a p-wff or a n-wff.

For conciseness, we introduce the following abbreviations:

Definition 5.9.

1. $\mathbb{K}A =_{df} \neg(A \to \bot)$
2. $\mathbb{P}A =_{df} (\neg A \to \bot)$

Given what has been said above, the following definition comes with no surprise:

Definition 5.10 (Knowledge in a state). *B is known in an M-state $\mathcal{S}$ iff $\mathcal{S} \looparrowright \mathbb{K}B$.*

Note that knowledge in a state is defined in terms of epistemic permittance. One can proceed analogously for the concept of epistemic possibility in a state.

Definition 5.11 (Epistemic possibility in a state). *B is epistemically possible in an M-state $\mathcal{S}$ iff $\mathcal{S} \looparrowright \mathbb{P}B$.*

Observe that for n-wffs, "being e-permittted by a state" and "being known is a state" coincide. This does not hold, however, for p-wffs. As for the latter, "being permitted by a state" and "being epistemically possible in a state" concide. Yet, in the case of n-wffs "being permitted by a state" and "being epistemically possible in the state" differ.

5.4.1 A Philosophical Comment

The standard philosophical concept of knowledge conceives it as a *true* justified belief about the actual world. In the framework of an epistemic logic supplemented with a relational semantics, "being known in a world w of a model" is explicated by "being true in each world w^* of the model such that w^* is accessible from w." When **S5** is used as an epistemic logic, this amounts to being true in each world of the model. Since we usually assume that the actual world is among the possible worlds considered (or is represented by a certain possible world of a model), the truth of $\Box B$ in a model yields the truth of B in the actual world, and $\Box B$ is true in the actual world only if B is true in the world.

Knowledge in a state behaves differently. If B is known in a state $\mathcal{S}$, it is true in each world of the state and thus also in the actual world *if* the actual world "is" in $\mathcal{S}$. This, however, need not be the case.

Note that knowledge in a state is defined neither in terms of belief nor justification (or any of Ω-generalizations of the latter). As a consequence, the problems faced by tripartite accounts of propositional knowledge, described in the previous chapter, do not show up in the current setting.

5.5 Epistemic Permittance and Inconsistency

As above, we assume that $M = \langle \mathcal{W}, \mathcal{V} \rangle$ is an arbitrary but fixed $\mathcal{L}$-model. Let X be a set of wffs of $\mathcal{L}$, and A be a wff of $\mathcal{L}$.

The *e-permittance class* of A in M, in symbols: $\|A\|_M$, comprises all the M-states that e-permit A. The *e-permittance class* of X in M, in symbols: $\|X\|_M$, is the intersection of the e-permittance classes of elements of X in M. More formally:

Definition 5.12 (e-permittance class)**.**

1. $\|A\|_M = \{\mathcal{S} \subseteq \mathcal{W} : \mathcal{S} \neq \emptyset$ *and* $\mathcal{S} \looparrowright A\}$.
2. $\|X\|_M = \{\mathcal{S} \subseteq \mathcal{W} : \mathcal{S} \looparrowright B$ *for each* $B \in X\}$.

Definition 5.13. *X has a non-empty e-permittance class iff there exists an $\mathcal{L}$-model M such that $\|X\|_M \neq \emptyset$.*

When $\{A\}$ has a non-empty e-permittance class, we will be saying briefly: "A has a non-empty e-permittance class."

One can show that some inconsistent sets of wffs have non-empty e-permittance classes. For clarity, let us first introduce:

Definition 5.14 (Inconsistent sets and plainly inconsistent sets)**.** *A set of wffs X of $\mathcal{L}$ is:*

1. *inconsistent iff* $\bigcap_{B \in X} |B|_M = \emptyset$ *for each $\mathcal{L}$-model M;*
2. *plainly inconsistent iff:*
   (a) *for some wff A, both $A \in X$ and '$\neg A$' $\in X$, or*
   (b) *for some wff $A \in X$, the singleton set $\{A\}$ is inconsistent.*

Clearly, e-permittance classes of plainly inconsistent sets are always empty. However, the situation is different in the case of some sets of wffs which are inconsistent, but not plainly inconsistent. For example, the set $\{A, A \to \bot\}$ is inconsistent. But the following holds:

Corollary 5.8. *Let A be a p-wff of $\mathcal{L}$ such that '$\Diamond A \to \Box A$' $\notin$ **S5**. Then there exists an $\mathcal{L}$-model M such that $\|\{A, A \to \bot\}\|_M \neq \emptyset$.*

Proof. When '$\Diamond A \to \Box A$' $\notin$ **S5**, there exists a **S5**-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ and a world $w \in \mathcal{W}$ such that $\mathcal{M}, w \models \Diamond A$ and $\mathcal{M}, w \models \Diamond \neg A$. So for some $w_1 \in \mathcal{W} : \mathcal{M}, w_1 \models A$, and for some $w_2 \in \mathcal{W} : \mathcal{M}, w_2 \models \neg A$. Consider the following $\mathcal{L}$-model M:

$$\langle \{w_1, w_2\}, \mathcal{V}|\{w_1, w_2\}\rangle$$

As both A and $(A \to \bot)$ are p-wffs, it is easily seen that for the state $\{w_1, w_2\}$ of the model we have $\{w_1, w_2\} \looparrowright A$ and $\{w_1, w_2\} \looparrowright (A \to \bot)$. Hence $\|\{A, (A \to \bot)\}\|_M \neq \emptyset$. □

In particular, the e-permittance class of $\{p, p \to \bot\}$ is non-empty.

Thus the following is true:

Corollary 5.9. *There exist: inconsistent sets of wffs of $\mathcal{L}$ and $\mathcal{L}$-models such that the sets have non-empty e-permittance classes in the models.*

Here is another example of an inconsistent set which has a non-empty e-permittance class.

Example 5.13. The set $\{p \to q, p, \neg q\}$ is inconsistent, but not plainly inconsistent. Let $M = \langle \mathcal{W}, \mathcal{V}\rangle$ be an $\mathcal{L}$-model such that for some $w_1, w_2 \in \mathcal{W}$:

- $\mathcal{V}(p, w_1) = \mathbf{0}$,
- $\mathcal{V}(q, w_1) = \mathbf{0}$,
- $\mathcal{V}(p, w_2) = \mathbf{1}$,
- $\mathcal{V}(q, w_2) = \mathbf{0}$.

Clearly we have:

- $M, w_2 \models p$ and hence $\{w_1, w_2\} \looparrowright p$,
- $M, w_1 \models (p \to q)$ and thus $\{w_1, w_2\} \looparrowright (p \to q)$,
- $M, w_1 \models \neg q$ as well as $M, w_2 \models \neg q$; therefore $\{w_1, w_2\} \looparrowright \neg q$.

Thus $\|\{p \to q, p, \neg q\}\|_M \neq \emptyset$.

5.6 Transmission of Epistemic Permittance

5.6.1 Definition and Basic Properties

Let us now introduce the concept of transmission of epistemic permittance between a set of wffs of $\mathcal{L}$ and a wff of $\mathcal{L}$. The symbol $\hookrightarrow_{\mathcal{L}}$ refers to the relation in question.

Definition 5.15 (Transmission of epistemic permittance)**.** *$X \hookrightarrow_{\mathcal{L}} A$ iff for each $\mathcal{L}$-model M and each M-state $\mathcal{S}$:*

$$\textit{if } \mathcal{S} \in \|X\|_M, \textit{ then } \mathcal{S} \in \|A\|_M.$$

The intuitive content of the above concept is: if all the elements of X are simultaneously e-permitted by a state, then A is e-permitted by the state. This condition is supposed to hold for each $\mathcal{L}$-model and each state of the model.

Let "$\mathcal{S} \looparrowright X$" abbreviate "for each $B \in X : \mathcal{S} \looparrowright B$."

Corollary 5.10. *$X \hookrightarrow_{\mathcal{L}} A$ iff the following condition:*

$$\textit{if } \mathcal{S} \looparrowright X, \textit{ then } \mathcal{S} \looparrowright A \tag{5.15}$$

is fulfilled by each state $\mathcal{S}$ of any $\mathcal{L}$-model.

Transmission of permittance, $\hookrightarrow_{\mathcal{L}}$, is a consequence relation. One can easily prove:

Corollary 5.11. *$\hookrightarrow_{\mathcal{L}}$ has the following properties:*

(Overlap) *If $A \in X$, then $X \hookrightarrow_{\mathcal{L}} A$.*

(Dilution) *If $X \hookrightarrow_{\mathcal{L}} A$ and $X \subseteq Y$, then $Y \hookrightarrow_{\mathcal{L}} A$.*

(Cut for sets) *If $X \cup Y \hookrightarrow_{\mathcal{L}} A$ and $X \hookrightarrow_{\mathcal{L}} B$ for every $B \in Y$, then $X \hookrightarrow_{\mathcal{L}} A$.*

Transmission of permittance, $\hookrightarrow_{\mathcal{L}}$, is not a structural consequence relation, however. The following examples illustrate this:[29]

Example 5.14.

$$\{\neg(p \wedge \neg q), p\} \hookrightarrow_{\mathcal{L}} q \tag{5.16}$$

To prove (5.16) suppose that for some state $\mathcal{S}$ of an $\mathcal{L}$-model M it holds that:

(1) $\mathcal{S} \looparrowright \neg(p \wedge \neg q)$, and

(2) $\mathcal{S} \looparrowright p$.

[29] For brevity, we use, here and below, object-level language expressions instead of their metalinguistic names.

By (2) there exists $w \in \mathcal{S}$, say, w_1, such that $M, w_1 \models p$. But since (1) holds as well, we have $M, w_1 \models \neg(p \wedge \neg q)$ and hence $M, w_1 \models q$. Thus $\mathcal{S} \looparrowright q$.

Example 5.15.

$$\{\neg(p \wedge \neg\neg q), p\} \not\hookrightarrow_{\mathcal{L}} \neg q \tag{5.17}$$

To see this it suffices to consider an $\mathcal{L}$-model $M = \langle \{w_1, w_2\}, \mathcal{V} \rangle$ in which $\mathcal{V}(p, w_1) = \mathbf{1}$, $\mathcal{V}(q, w_1) = \mathbf{0}$, $\mathcal{V}(p, w_2) = \mathbf{0}$, and $\mathcal{V}(q, w_2) = \mathbf{1}$. We get:

- $M, w_1 \models p$,
- $M, w_1 \models \neg(p \wedge \neg\neg q))$,
- $M, w_2 \models \neg(p \wedge \neg\neg q))$

Thus $\{w_1, w_2\} \looparrowright \{\neg(p \wedge \neg\neg q), p\}$. On the other hand, since $M, w_1 \not\models \neg q$, we have $\{w_1, w_2\} \not\looparrowright \neg q$.

Generally speaking, $\hookrightarrow_{\mathcal{L}}$ is not structural because substitution can change the categories of wffs, that is, can turn p-wffs into n-wffs, or n-wffs into p-wffs.[30]

5.6.2 Transmission of Epistemic Permittance versus Entailment

Entailment in $\mathcal{L}$, $\models_{\mathcal{L}}$, can be defined by:

Definition 5.16 (Entailment in $\mathcal{L}$). *$X \models_{\mathcal{L}} A$ iff for each $\mathcal{L}$-model M:*

$$\bigcap_{B \in X} |B|_M \subseteq |A|_M$$

Entailment in $\mathcal{L}$ amounts to entailment determined by Classical Propositional Logic.

Transmission of e-permittance is a special case of entailment. By Corollary 5.6 we get:

Corollary 5.12. *If $X \hookrightarrow_{\mathcal{L}} A$, then $X \models_{\mathcal{L}} A$.*

Proof. Let M be an arbitrary but fixed $\mathcal{L}$-model. Assume that all the wffs in X are simultaneously true in a world w of M. By Corollary 5.6, it follows that $\{w\} \looparrowright X$. Since, by assumption, $X \hookrightarrow_{\mathcal{L}} A$, it follows that $\{w\} \looparrowright A$. Therefore, by Corollary 5.6 again, A is true in the world w of M. Thus $X \models_{\mathcal{L}} A$. □

[30] This can happen when the wff being substituted is a propositional variable or has the form $\neg \ldots \neg \boldsymbol{p}$, where $\boldsymbol{p}$ is a propositional variable.

Hence $\hookrightarrow_{\mathcal{L}}$ is a *truth-preserving* consequence relation.

The converse of Corollary 5.12 does not hold. The following examples illustrate this:

Example 5.16.

$$\neg p \vee \neg q \not\hookrightarrow_{\mathcal{L}} \neg(p \wedge q) \tag{5.18}$$

For, consider an $\mathcal{L}$-model $M = \langle \{w_1, w_2\}, \mathcal{V} \rangle$ such that $\mathcal{V}(p, w_1) = \mathbf{0}$, $\mathcal{V}(p, w_2) = \mathbf{1}$, and $\mathcal{V}(q, w_2) = \mathbf{1}$. Since $\neg p \vee \neg q$ is a p-wff, $\{w_1, w_2\} \looparrowright \neg p \vee \neg q$. On the other hand, $\neg(p \wedge q)$ is a n-wff and we have $\{w_1, w_2\} \not\looparrowright \neg(p \wedge q)$ because $M, w_2 \models (p \wedge q)$.

Example 5.17.

$$\{p \rightarrow q, \neg q\} \not\hookrightarrow_{\mathcal{L}} \neg p \tag{5.19}$$

To see this it suffices to consider an $\mathcal{L}$-model $M = \langle \{w_1, w_2\}, \mathcal{V} \rangle$ in which $\mathcal{V}(p, w_1) = \mathbf{0}$, $\mathcal{V}(q, w_1) = \mathbf{0}$, $\mathcal{V}(p, w_2) = \mathbf{1}$, and $\mathcal{V}(q, w_2) = \mathbf{0}$. Since $M, w_1 \models (p \rightarrow q)$, we get $\{w_1, w_2\} \looparrowright (p \rightarrow q)$. Clearly, $\{w_1, w_2\} \looparrowright \neg q$. But $\{w_1, w_2\} \not\looparrowright \neg p$ because $\mathcal{V}(p, w_2) = \mathbf{1}$.

5.6.3 Paraconsistency

As we have shown in section 5.5, some inconsistent sets have non-empty permittance classes. It follows that $\hookrightarrow_{\mathcal{L}}$ is *paraconsistent* in the following sense of the word: it is not the case that for every inconsistent set X and every wff B it holds that $X \hookrightarrow_{\mathcal{L}} B$.

Example 5.18. The set $\{p \rightarrow q, p, \neg q\}$ has a non-empty e-permittance class (see Example 5.13). Hence, in particular:

$$\{p \rightarrow q, p, \neg q\} \not\hookrightarrow_{\mathcal{L}} r \tag{5.20}$$

Example 5.19. The set $\{p, p \rightarrow \bot\}$ is inconsistent, but has a non-empty e-permittance class. One can easily show that:

$$\{p, p \rightarrow \bot\} \not\hookrightarrow_{\mathcal{L}} q \tag{5.21}$$

Observe, however, that we still have:

$$\{p, \neg p\} \hookrightarrow_{\mathcal{L}} q \tag{5.22}$$

5.7 A Checking Method

How can one check whether transmission of e-permittance, $\hookrightarrow_{\mathcal{L}}$, holds between a given set of wffs of $\mathcal{L}$ and a given wff of $\mathcal{L}$? In this section we present a solution to this problem.

5.7.1 Translation ( )*

Let us first define an operation, $(\ )^*$, which assigns to a wff of $\mathcal{L}$ the corresponding wff of $\mathfrak{L}$.

Definition 5.17 (Translation ( )*).

1. *If A is a p-wff of $\mathcal{L}$, then $(A)^* = \Diamond A$.*
2. *If A is a n-wff of $\mathcal{L}$, then $(A)^* = \Box A$.*

Note that A in $\Diamond A$ or in $\Box A$ represents a wff of the ("non-modal") language $\mathcal{L}$. The operation $(\ )^*$ is performed on A only once; the subformulas of A remain unaffected. In other words, $(\ )^*$ is a kind of "surface translation" of wffs of $\mathcal{L}$ into wffs of $\mathfrak{L}$.[31]

For convenience, we put:

$$(X)^* =_{df} \{(A)^* : A \in X\}$$

Let us now prove:

Lemma 5.3. *If $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ is a* **S5***-model such that $\mathcal{M} \models (X)^*$, then $\mathcal{W} \looparrowright X$.*

Proof. First observe that $\mathcal{M}$ is the **S5**-model accompanied with an $\mathcal{L}$-model $M = \langle \mathcal{W}, \mathcal{V} \rangle$ w.r.t. the M-state $\mathcal{W}$.

The elements of $(X)^*$ are either of the form $\Diamond B$ or of the form $\Box B$, where $B \in X$. More precisely, if B is a p-wff, then the corresponding element of $(X)^*$ is $\Diamond B$, and if B is a n-wff, $\Box B$ is the corresponding element of $(X)^*$.

When $\Diamond B \in (X)^*$ and $\mathcal{M} \models \Diamond B$, we get $\mathcal{W} \looparrowright B$ by Lemma 5.1.

The case in which $\Box B \in (X)^*$ is analogous. □

The following holds:

Theorem 5.1. *X has a non-empty e-permittance class iff there exists a* **S5***-model $\mathcal{M}$ such that $\mathcal{M} \models (X)^*$.*

Proof. (⇒). Let M be an $\mathcal{L}$-model for which $\|X\|_M \neq \emptyset$. Let $\mathcal{S} \in \|X\|_M$. We consider the **S5**-model $\mathcal{M}_{\mathcal{S}}$ accompanied with M w.r.t. $\mathcal{S}$, and we apply Lemma 5.1.
(⇐). By Lemma 5.3. □

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[31] The idea of using translations into **S5** in constructing paraconsistent logics goes back to Jaśkowski (cf. [24], and [25] for an English translation). However, Jaśkowski's translation is defined recursively and enables an introduction of "discussive" connectives. The operation ( )* behaves differently.

Example 5.20. As we have shown (see Example 5.13), the inconsistent set $\{p \to q, p, \neg q\}$ has a non-empty e-permittance class. The following takes place on the modal side:

$$\mathcal{M}_{\{w_1,w_2\}} \models \{\Diamond(p \to q), \Diamond p, \Box\neg q\} \tag{5.23}$$

where $\mathcal{M}_{\{w_1,w_2\}}$ is the **S5**-model accompanied (w.r.t. state $\{w_1, w_2\}$) with the $\mathcal{L}$-model considered in Example 5.13.

However, the following holds:

Corollary 5.13. *If X is inconsistent and each element of X is a n-wff, then the e-permittance class of X is empty.*

Proof. Suppose that the e-permittance class of X is non-empty. Then, by Theorem 5.1, for some **S5**-model $\mathcal{M}$ we have $\mathcal{M} \models (X)^*$. But the elements of $(X)^*$ are of the form $\Box A$, where $A \in X$. Since $\mathcal{W}$ is non-empty, there exists a world w of $\mathcal{M}$ such that $\mathcal{M}, w \models X$. It follows that X is consistent. □

The situation can be different when X contains some p-wffs.

5.7.2 Transmission of Epistemic Permittance versus Global S5-entailment

Recall that Φ stands for a set of wffs of $\mathfrak{L}$ (i.e. the modal extension of $\mathcal{L}$), and ϕ is a metalanguage variable for wffs of $\mathfrak{L}$.

Let us introduce:

Definition 5.18 (Global **S5**-entailment)**.** $\Phi \models_{\mathbf{S5}} \phi$ *iff for each* **S5***-model* $\mathcal{M}$*: if* $\mathcal{M} \models \Phi$*, then* $\mathcal{M} \models \phi$*.*

We will now prove:

Theorem 5.2 (Reduction modulo $(\ )^*$)**.** $X \hookrightarrow_{\mathcal{L}} A$ *iff* $(X)^* \models_{\mathbf{S5}} (A)^*$*.*

Proof. Suppose that $X \hookrightarrow_{\mathcal{L}} A$, but $(X)^* \not\models_{\mathbf{S5}} (A)^*$. Thus for some **S5**-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ we have $\mathcal{M} \models (X)^*$ and $\mathcal{M} \not\models (A)^*$. But $\mathcal{M}$ is accompanied with the $\mathcal{L}$-model $M = \langle \mathcal{W}, \mathcal{V} \rangle$ w.r.t. $\mathcal{W}$, that is, $\mathcal{M} = \mathcal{M}_{\mathcal{W}}$. By Lemma 5.3 we get $\mathcal{W} \looparrowright X$ and hence, due to the transmission of e-permittance, $\mathcal{W} \looparrowright A$. If A is a p-wff, then, by Lemma 5.1, $\mathcal{M} \models \Diamond A$, that is, $\mathcal{M} \models (A)^*$. A contradiction. Similarly, if A is a n-wff, by Lemma 5.1 we get $\mathcal{M} \models \Box A$, i.e. $\mathcal{M} \models (A)^*$. A contradiction again.

Now suppose that $(X)^* \models_{\mathbf{S5}} (A)^*$, but $X \not\hookrightarrow_{\mathcal{L}} A$. Then there exists a state $\mathcal{S}$ of a certain $\mathcal{L}$-model M such that $\mathcal{S} \hookrightarrow X$ and $\mathcal{S} \not\hookrightarrow A$. We consider the **S5**-model $\mathcal{M}_{\mathcal{S}}$ accompanied with M w.r.t. $\mathcal{S}$. By Lemma 5.1 we get $\mathcal{M}_{\mathcal{S}} \models (X)^*$ and $\mathcal{M}_{\mathcal{S}} \not\models (A)^*$. A contradiction. □

According to Theorem 5.2, transmission of e-permittance amounts to (global) **S5**-entailment among the relevant *-wffs. Thus in order to check whether the transmission of e-permittance holds between wffs of $\mathcal{L}$, it suffices to check whether global **S5**-entailment takes place between the corresponding *-wffs

Remark 5.4. Theorem 5.2 does not say that transmission of e-permittance can be *identified with* global **S5**-entailment. Recall that the *-wffs are either of the form $\Box A$ or of the form $\Diamond A$, where A is a wff of the non-modal language $\mathcal{L}$ (and thus does not involve modal operators).

Necessity and possibility are, in a sense, expressible in $\mathcal{L}$ (cf. section 5.3). But when we have $\phi \models_{\mathbf{S5}} \psi$ for $\mathfrak{L}$-wffs ϕ, ψ which are of neither of the forms: $\Box A$, $\Diamond A$, the systematic replacement in ϕ and ψ of $\Box A$ by $\neg(A \to \bot)$ as well as of $\Diamond A$ by $(\neg A \to \bot)$ need not turn global **S5**-entailment between ϕ and ψ into the transmission of e-permittance between the resultant wffs of $\mathcal{L}$.

Example 5.21. We have:

$$\neg\Box p \models_{\mathbf{S5}} \Box\neg\Box p \tag{5.24}$$

By the systematic replacement we get:

$$\neg\neg(p \to \bot) \hookrightarrow_{\mathcal{L}} \neg(\neg\neg(p \to \bot) \to \bot) \tag{5.25}$$

(5.25) *does not* hold, however. To see this let us take an $\mathcal{L}$-model $M^* = \langle\{w_1, w_2\}, \mathcal{V}\rangle$ such that $\mathcal{V}(p, w_1) = \mathbf{0}$ and $\mathcal{V}(p, w_2) = \mathbf{1}$. Clearly, we have:

$$\{w_1, w_2\} \hookrightarrow \neg\neg(p \to \bot) \tag{5.26}$$

since $M^*, w_1 \models \neg\neg(p \to \bot)$. At the same time we have:

$$\{w_1, w_2\} \not\hookrightarrow \neg(\neg\neg(p \to \bot) \to \bot) \tag{5.27}$$

because $M^*, w_2 \not\models \neg(\neg\neg(p \to \bot) \to \bot)$.

5.8 What is Retained and What is Lost

5.8.1 The Case of Single Wffs

Let us first prove:

Corollary 5.14. *If* $B \models_{\mathcal{L}} A$ *and*

1. *B and A are p-wffs, or*
2. *B and A are n-wffs, or*
3. *B is a n-wff and A is a p-wff,*

then $B \hookrightarrow_{\mathcal{L}} A$.

Proof. If $B \models_{\mathcal{L}} A$, then $\models_{\mathcal{L}} (B \to A)$ and hence '$\Box(B \to A)$' $\in$ **S5**.

Assume that B and A are p-wffs. Suppose that $\Diamond B \not\models_{\mathbf{S5}} \Diamond A$. So there exists a **S5**-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ such that $\mathcal{M} \models \Diamond B$ and $\mathcal{M} \not\models \Diamond A$. Hence $\mathcal{M}, w \not\models A$ for each $w \in \mathcal{W}$, and $\mathcal{M}, w \models B$ for some $w \in \mathcal{W}$. It follows that for some $w \in \mathcal{W}$ we have $\mathcal{M}, w \not\models (B \to A)$ and therefore '$\Box(B \to A)$'$\notin$ **S5**. A contradiction. Thus $\Diamond B \models_{\mathbf{S5}} \Diamond A$ and hence, by Theorem 5.2, $B \hookrightarrow_{\mathcal{L}} A$.

Assume that B and A are n-wffs. Suppose that $\Box B \not\models_{\mathbf{S5}} \Box A$. So for some **S5**-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ we get: $\mathcal{M}, w \models B$ for any $w \in \mathcal{W}$, and $\mathcal{M}, w \not\models A$ for some $w \in \mathcal{W}$. Thus '$\Box(B \to A)$' $\notin$ **S5**. A contradiction. Therefore, by Theorem 5.2, $B \hookrightarrow_{\mathcal{L}} A$.

Finally, assume that B is a n-wff and A is a p-wff. Suppose that $\Box B \not\models_{\mathbf{S5}} \Diamond A$. Thus, for some **S5**-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$, $\mathcal{M}, w \models B$ for any $w \in \mathcal{W}$, and $\mathcal{M}, w \not\models A$ for each $w \in \mathcal{W}$. Hence '$\Box(B \to A)$'$\notin$ **S5**. A contradiction again. Therefore, by Theorem 5.2, $B \hookrightarrow_{\mathcal{L}} A$. □

Thus, for instance, the following hold:[32]

$$p \hookrightarrow_{\mathcal{L}} \neg\neg p \tag{5.28}$$

$$\neg\neg p \hookrightarrow_{\mathcal{L}} p \tag{5.29}$$

$$(p \to q) \hookrightarrow_{\mathcal{L}} (\neg q \to \neg p) \tag{5.30}$$

$$(\neg q \to \neg p) \hookrightarrow_{\mathcal{L}} (p \to q) \tag{5.31}$$

[32] Recall that $\hookrightarrow_{\mathcal{L}}$ is not a sentential connective, but a sign of a relation that holds between wffs. As above, for brevity we use object-language expressions instead of their metalinguistic names.

$$p \hookrightarrow_{\mathcal{L}} (q \rightarrow p) \tag{5.32}$$

$$(p \rightarrow q) \wedge p \hookrightarrow_{\mathcal{L}} q \tag{5.33}$$

$$(p \vee q) \wedge \neg q \hookrightarrow_{\mathcal{L}} p \tag{5.34}$$

$$(p \vee \neg q) \wedge q \hookrightarrow_{\mathcal{L}} p \tag{5.35}$$

$$(p \rightarrow (q \rightarrow r)) \hookrightarrow_{\mathcal{L}} ((p \rightarrow q) \rightarrow (p \rightarrow r)) \tag{5.36}$$

$$(p \rightarrow (q \rightarrow r)) \hookrightarrow_{\mathcal{L}} (p \wedge q \rightarrow r) \tag{5.37}$$

$$(p \wedge q \rightarrow r) \hookrightarrow_{\mathcal{L}} (p \rightarrow (q \rightarrow r)) \tag{5.38}$$

$$(p \rightarrow (q \rightarrow r)) \hookrightarrow_{\mathcal{L}} (q \rightarrow (p \rightarrow r)) \tag{5.39}$$

$$((p \rightarrow q) \wedge (q \rightarrow r)) \hookrightarrow_{\mathcal{L}} (p \rightarrow r) \tag{5.40}$$

$$\neg(p \wedge q) \hookrightarrow_{\mathcal{L}} (\neg p \vee \neg q) \tag{5.41}$$

$$\neg(p \vee q) \hookrightarrow_{\mathcal{L}} (\neg p \wedge \neg q) \tag{5.42}$$

$$\neg(p \wedge \neg q) \hookrightarrow_{\mathcal{L}} (p \rightarrow q) \tag{5.43}$$

$$\neg(p \rightarrow q) \hookrightarrow_{\mathcal{L}} (p \wedge \neg q) \tag{5.44}$$

Observe, however, that the converses of (5.41), (5.42), (5.43) and (5.44) do not hold. The counterpart of *Modus Tollendo Tollens* does not hold either, i.e.:

$$((p \rightarrow q) \wedge \neg q) \not\hookrightarrow_{\mathcal{L}} \neg p \tag{5.45}$$

because:

$$\Diamond((p \rightarrow q) \wedge \neg q) \not\models_{\mathbf{S5}} \Box\neg p \tag{5.46}$$

Hence:

Corollary 5.15. *There are cases in which: B is a p-wff, A is a n-wff, $B \models_{\mathcal{L}} A$, and $B \not\hookrightarrow_{\mathcal{L}} A$.*

Yet, the following holds:

$$((p \rightarrow q) \wedge \neg q) \hookrightarrow_{\mathcal{L}} \mathbb{P}\neg p \tag{5.47}$$

(Recall that $\mathbb{P}\neg p$ claims that $\neg p$ is epistemically possible in a state.) This can be generalized.

Corollary 5.16. *If $B \models_{\mathcal{L}} A$ and*

1. *B is a p-wff and*

2. A is a n-wff,

then $B \hookrightarrow_{\mathcal{L}} \mathbb{P}A$.

Proof. If $B \models_{\mathcal{L}} A$, then '$\Box(B \to A)$'$\in$ **S5**. Suppose that $\Diamond B \not\models_{\mathbf{S5}} \Diamond\mathbb{P}A$. So for some **S5**-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V}\rangle$ there exists $w_1 \in \mathcal{W}$ such that $\mathcal{M}, w_1 \models B$ and, at the same time, $\mathcal{M}, w \not\models \mathbb{P}A$ for any $w \in \mathcal{W}$. Recall that $\mathbb{P}A =_{df} (\neg A \to \bot)$. Hence for each $w \in \mathcal{W}$ we have $\mathcal{M}, w \not\models A$. Therefore '$\Box(B \to A)$'$\notin$ **S5**. A contradiction. □

5.8.2 The Case of Sets of Wffs

The direct counterpart of *Modus Ponens* holds for $\hookrightarrow_{\mathcal{L}}$ (cf. 5.33). But we have:[33]

$$\{p \to q, p\} \not\hookrightarrow_{\mathcal{L}} q \tag{5.48}$$

So conjunction behaves is a non-standard way in the context of $\hookrightarrow_{\mathcal{L}}$: $A_1 \wedge \ldots \wedge A_n \hookrightarrow_{\mathcal{L}} B$ need not be tantamount to $\{A_1, \ldots, A_n\} \hookrightarrow_{\mathcal{L}} B$. The reason is that a e-permittance class of a set of wffs need not be equal with the e-permittance class of a conjunction of all the wffs in the set.[34]

Yet, the following is true:

$$\{p \to q, \mathbb{K}p\} \hookrightarrow_{\mathcal{L}} q \tag{5.49}$$

Recall that $\mathbb{K}p$ can be read: "p is known in a state in question."

Here are further "negative" examples:

$$\{p, q\} \not\hookrightarrow_{\mathcal{L}} (p \wedge q) \tag{5.50}$$

$$\{p, p \to \bot\} \not\hookrightarrow_{\mathcal{L}} (p \wedge \neg p) \tag{5.51}$$

$$\{p \vee \neg q, q\} \not\hookrightarrow_{\mathcal{L}} p \tag{5.52}$$

$$\{p \to q, q \to r\} \not\hookrightarrow_{\mathcal{L}} (p \to r) \tag{5.53}$$

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[33] Since $\{\Diamond(p \to q), \Diamond p\} \not\models_{\mathbf{S5}} \Diamond q$. (5.33) holds because $\Diamond((p \to q) \wedge p) \models_{\mathbf{S5}} \Diamond q$.

[34] For example, take an $\mathcal{L}$-model $M = \langle \{w_1, w_2\}, \mathcal{V}\rangle$ such that $\mathcal{V}(p, w_1) = \mathbf{0}$, $\mathcal{V}(q, w_1) = \mathbf{0}$, $\mathcal{V}(p, w_2) = \mathbf{1}$, and $\mathcal{V}(q, w_2) = \mathbf{0}$. Clearly, $\{w_1, w_2\} \in \|(p \to q), p\|_M$, but $\{w_1, w_2\} \notin \|(p \to q) \wedge p\|_M$. In general, a conjunction of p-wffs carries information that the conjuncts are simultaneously true in some world(s) of a state, while the information carried by the set of conjuncts amounts to the claim that each conjunct is true in a certain world of the state. When we have a "mixed" conjunction (that is, involving both p-wff and n-wffs), the information carried by n-wffs "weakens": the consecutive conjuncts, n-wffs included, are supposed to simultaneously hold in a certain world of a state.

Observe, however, that the following hold:

$$\{\mathbb{K}p, q\} \hookrightarrow_{\mathcal{L}} (p \wedge q) \tag{5.54}$$

and similarly for q,

$$\{\mathbb{K}p, \mathbb{K}q\} \hookrightarrow_{\mathcal{L}} \mathbb{K}(p \wedge q) \tag{5.55}$$

$$\{p \vee \neg q, \mathbb{K}q\} \hookrightarrow_{\mathcal{L}} p \tag{5.56}$$

$$\{\neg(\neg p \wedge q), q\} \hookrightarrow_{\mathcal{L}} p \tag{5.57}$$

$$\{\mathbb{K}(p \to q), \mathbb{K}(q \to r)\} \hookrightarrow_{\mathcal{L}} \mathbb{K}(p \to r) \tag{5.58}$$

$$\{\neg(p \wedge \neg q), \neg(q \wedge \neg r)\} \hookrightarrow_{\mathcal{L}} \neg(p \wedge \neg r) \tag{5.59}$$

It happens that conjunction behaves in the "standard" way in the context of $\hookrightarrow_{\mathcal{L}}$ although the conjuncts belong to diverse categories, as in:

$$\{p \to q, \neg(q \wedge \neg r)\} \hookrightarrow_{\mathcal{L}} (p \to r) \tag{5.60}$$

$$\{\neg p \to q, \neg p\} \hookrightarrow_{\mathcal{L}} q \tag{5.61}$$

$$\{p \vee q, \neg q\} \hookrightarrow_{\mathcal{L}} p \tag{5.62}$$

$$\{\neg p \to q, \neg q\} \hookrightarrow_{\mathcal{L}} p \tag{5.63}$$

Let us now turn to inconsistent sets. As we have shown, $\hookrightarrow_{\mathcal{L}}$ is paraconsistent. But, for instance, we still have:

$$\{p \to q, p, \neg q\} \hookrightarrow_{\mathcal{L}} (\neg p \vee q) \tag{5.64}$$

$$\{p \to q, p, \neg q\} \hookrightarrow_{\mathcal{L}} (\neg(p \to q) \vee \neg p) \tag{5.65}$$

$$\{p \to q, p, \neg q\} \hookrightarrow_{\mathcal{L}} ((\neg(p \to q) \vee \neg p) \vee q) \tag{5.66}$$

$$\{r, s, (r \to p), (s \to \neg p)\} \hookrightarrow_{\mathcal{L}} (p \vee \neg p) \tag{5.67}$$

$$\{r, s, \mathbb{K}(r \to p), \mathbb{K}(s \to \neg p)\} \hookrightarrow_{\mathcal{L}} (\mathbb{K}r \vee \mathbb{K}s) \tag{5.68}$$

5.9 Final Remarks

Epistemic permittance has been defined here as a semantic relation between a wff and a set of possible worlds, where – and this is crucial from the intuitive point of view – the possible worlds are supposed to represent alternative accounts of how things are. In accordance with the videly accepted terminological convention, sets of possible worlds were called states. It should be stressed, however, that states in our sense are not thought of as information states of an agent.

The concepts of being known in a state and being epistemically possible in a state can be defined in terms of e-permittance by a state. Their scopes differ from that of e-permittance, and differ among themselves. The respective concepts of knowledge and epistemic possibility are non-standard in the sense that they are relativized to states. Moreover, knowledge and epistemic possibility are not conceived here as propositional attitudes held (or not) by cognitive agents. Epistemic permittance is not conceived that way either.

Our analysis was pursued on the semantic level, and the language for which e-permittance (and related notions) were defined, was non-modal: its syntax did not contain operators representing e-permittance, knowledge and epistemic possibility. A step towards a proof-theoretic account would require introduction of such operators and possibly a connective by which transmission of e-permittance can be expressed.

The concept of e-permittance was used in [59] for modeling question raising by inconsistent premises. However, its range of applicability is not restricted to the logic of questions. It is an open question how wide the range is.

The last remark is this. The relativization of knowledge to states seems to resolve the old philosophical problem: one can legitimately claim that A is an item of knowledge in some initial state and ceases to constitute knowledge as the initial state is enriched with a new possible world/a new account of how things are in which A is not true anymore.[35] Moreover, let us consider the case of conflicting hypotheses being general statements of the form $\forall x_i \mathsf{A}$. Assuming that they are treated semantically as we have treated p-wffs, conflicting hypotheses can be simultaneously e-permitted by a state and this is not tantamount to falling into a contradiction. A hypothesis of this kind constitutes an item of knowledge in a state if it is true in *each* world of the state, and

[35] More precisely, A ceases to constitute knowledge with respect to the "new" state.

extending the state with a new world in which the claim of the hypothesis does not hold only changes its epistemic status, but does not require the rejection of the hypothesis: it remains an item of knowledge in the “old” state and becomes (only) e-permitted by the “new” state. E-permitted n-wffs, in turn, perform the role of *state-constraints*, since in their case e-permittance by a state equals being true in each world of the state.

Part III

Specifying

Chapter 6

Strong Multiple-Conclusion Entailment

6.1 Introductory Remarks

In this chapter I introduce and examine a concept of multiple-conclusion entailment, which I dub "strong multiple-conclusion entailment." Formally, strong mc-entailment is a subrelation of mc-entailment. Strong mc-entailment is defined in a way which allows us to avoid some drawbacks of the "standard" mc-entailment which are indicated below. Moreover, strong mc-entailment is neither left-monotone nor right-monotone. As a by-product one gets a concept of single-conclusion entailment dubbed "strong single-conclusion entailment," which, in turn, is free of the drawbacks pointed out below. This concept will be defined and examined in the next chapter.

6.1.1 Single-Conclusion Entailment: Drawbacks of the Received View

The idea of *transmission of truth* underlies the intuitive concept of entailment. According to the idea, entailment is akin to an input-output device which, when fed with truth at the input, gives truth at the output. The input need not consist of truths, but *if* it does, it transforms into a true output. Similarly, *if* the premises are all true, any conclusion entailed by them must be true, although the truth of premises is not a necessary condition for entailment to hold. Or, to put it differently, the *hypothetical* truth of premises warrants the truth of an entailed conclusion.[36]

Logicians operate with well-formed formulas (*wffs* for short) of formalized languages and conceptualize entailment as a semantic relation between sets of wffs and single wffs. At the same time they tend to understand the "if" above in the sense of material conditional. Yet, since a material conditional with false antecedent is true irrespective of the logical value of the consequent, as a consequence one gets:

(I) *a set of wffs which cannot be simultaneously true, i.e. an inconsistent set, entails every wff.*

Moreover, a material conditional with true consequent is true irrespective of the logical value of the antecedent, and hence:

(II) *a logically valid wff is entailed by any set of wffs.*

Both (I) and (II) are a kind of by-products and we got accustomed to live with them. But (I) as well as (II) seem to contravene the intuitive idea of transmission of truth. To say that "truth is transmitted" seems to presuppose that it *can* occur at the input and that it *need not* occur at the output.

Another drawback of the received view is this. To say that the hypothetical truth of sentences in a set X warrants the truth of a sentence B seems to presuppose that the hypothetical truth of *all* the sentences in X contributes to the hypothetical truth of B. Entailment intuitively construed is a kind of semantic entrenchment of an entailed sentence in a set of sentences that entails it: a set of sentences X that entails a sentence B comprises neither less nor more sentences than those the hypothetical truth of which, jointly, warrants the truth of B. On the other hand, entailment defined in the usual way, by using, inter alia, the material "if," is monotone:

(M) *a wff B entailed by a set of wffs X is entailed by any superset of X as well*

and hence the wff B is also entailed by sets of wffs which contain elements that are irrelevant with regard to the transmission of truth and/or the semantic entrenchment effect(s): their hypothetical truth do not contribute in any way to the truth of B.

[36] The latter statement can be explicated as: "If the truth-conditions of all the premises are met, an entailed conclusion is true as well."

6.1.2 Multiple-Conclusion Entailment: Drawbacks of the Received View

The concept of entailment is sometimes generalized to the concept of multiple-conclusion entailment (*mc-entailment* for short). Mc-entailment is a semantic relation between sets of wffs, where an entailed set is allowed to contain more than one element. The underlying idea is: an mc-entailed set must contain at least one true wff *if* the respective mc-entailing set consists of truths. Or, to put it differently, the hypothetical truth of all the wffs in an mc-entailing set warrants the existence of a true wff in the mc-entailed set

Mc-entailment can hold for trivial reasons: X mc-entails Y because X single-conclusion entails (*sc-entails* for short) at least one wff in Y. But mc-entailment can also hold non-trivially: it happens that a set of wffs, X, mc-entails a set of wffs, Y, although X does not sc-entail any wff in Y. For instance (taking Classical Propositional Logic as the basis), the truth of all the wffs in the set $X = \{p \rightarrow q \vee r, p\}$ warrants the existence of a true wff in the set $Y = \{q, r\}$, but neither q nor r is sc-entailed by X or, to put it differently, the hypothetical truth of the wffs in X guarantees that at least one of: q, r, is true, but warrants neither the truth of q nor the truth of r.

The concept of mc-entailment is more general than that of sc-entailment. One can always define sc-entailment as mc-entailment of a singleton set. However, it is not the case that mc-entailment can always be defined in terms of sc-entailment.[37]

One of the ways of thinking of entailed non-singleton *sets* is to construe them as items effectively *delimiting* search spaces: a set of wffs Y entailed by a set of wffs X is a minimal set that comprises wffs among which a truth must lie if the wffs in X are all true. "Minimal" means here "no proper subset of Y behaves analogously w.r.t. X." Another way of thinking about an entailed set is to construe it as characterizing the *relevant cases* to be considered, for if X mc-entails Y and each wff in Y sc-entails a wff B, the wff B is sc-entailed by X as well. However, the standard concept of mc-entailment is too broad to reflect the above ideas. This is due to the fact that mc-entailment is *right-monotone*.

(RM) *if a set of wffs X mc-entails a set of wffs, Y, then X mc-entails any superset of Y as well.*

[37] Cf. Chapter 1, section 1.1.

Observe also that mc-entailment explicated by means of the material "if" suffers similar drawbacks to those of sc-entailment explicated in this way:

(I') *any set of wffs is mc-entailed by an inconsistent set of wffs,* and

(II') *a set of wffs that contains a logically valid wff is mc-entailed by any set of wffs.*

Moreover, mc-entailment is *left-monotone*, that is:

(LM) *a set of wffs Y which is mc-entailed by a set of wffs X is also mc-entailed by any superset of X.*

Hence there exist mc-entailing sets of wffs which contain, inter alia, wffs that are semantically irrelevant to the corresponding mc-entailed sets. For example, $\{s, p \to q \vee r, p\}$ mc-entails $\{q, r\}$, while the hypothetical truth of s is completely irrelevant to the occurrence of truth in $\{q, r\}$.

6.2 The Logical Basis

For simplicity, we remain at the propositional level, and we consider the case of Classical Propositional Logic (hereafter: CPL). We assume that CPL is expressed in a language characterized as follows.

The vocabulary of the language comprises a countably infinite set `Var` of propositional variables, the connectives: $\neg, \vee, \wedge, \to$, and brackets. The set `Form` of *well-formed formulas* (wffs) of the language is the smallest set that includes `Var` and satisfies the following conditions: (1) if $A \in$ `Form`, then '$\neg A$' $\in$ `Form`; (2) if $A, B \in$ `Form`, then '$(A \otimes B)$' $\in$ `Form`, where $\otimes$ is any of the connectives: $\vee, \wedge, \to$. We adopt the usual conventions concerning omitting brackets. We use A, B, C, D, with subscripts when needed, as metalanguage variables for wffs, and X, Y, W, Z, with or without subscripts or superscripts, as metalanguage variables for sets of wffs. The letters p, q, r, s, t are exemplary elements of `Var`.

By a proper superset of a set of wffs X we mean a set of wffs Z such that X is a proper subset of Z.

For the sake of brevity, we adopt the following notational conventions:

- we write X, Y instead of $X \cup Y$,
- X, A abbreviates $X \cup \{A\}$,

- $X_{\ominus A}$ abbreviates $X \setminus \{A\}$.

These conventions will be applied as long as there is no risk of a misunderstanding.

The inscriptions $\bigwedge X$ and $\bigvee Y$ refer to a conjunction of all the wffs in a non-empty and finite set of wffs X and to a disjunction of all the wffs in X, respectively. If X is a singleton set, $\{A\}$, then $\bigwedge X = \bigvee X = A$.

Let $\mathbf{1}$ stand for truth and $\mathbf{0}$ for falsity. A CPL-*valuation* is a function $v : \mathtt{Form} \mapsto \{\mathbf{1}, \mathbf{0}\}$ satisfying the following conditions: (a) $v(\neg A) = \mathbf{1}$ iff $v(A) = \mathbf{0}$; (b) $v(A \vee B) = \mathbf{1}$ iff $v(A) = \mathbf{1}$ or $v(B) = \mathbf{1}$; (c) $v(A \wedge B) = \mathbf{1}$ iff $v(A) = \mathbf{1}$ and $v(B) = \mathbf{1}$; (d) $v(A \rightarrow B) = \mathbf{1}$ iff $v(A) = \mathbf{0}$ or $v(B) = \mathbf{1}$. Remark that the domain of v includes `Var`.

For brevity, in what follows we will be omitting references to CPL. Unless otherwise stated, the semantic relations analysed are supposed to hold between sets of CPL-wffs, or sets of CPL-wffs and single CPL-wffs. By valuations we will mean CPL-valuations.

In order to make this chapter self-contained let me again introduce the following notions (some of them were already introduced in Chapter 3).[38]

Definition 6.1 (Single-conclusion entailment; sc-entailment)**.** $X \models A$ *iff for each valuation* v*:*

- *if* $v(B) = \mathbf{1}$ *for every* $B \in X$*, then* $v(A) = \mathbf{1}$.

Definition 6.2 (Multiple-conclusion entailment; mc-entailment)**.** $X \parallel\!\models Y$ *iff for each valuation* v*:*

- *if* $v(B) = \mathbf{1}$ *for every* $B \in X$*, then* $v(A) = \mathbf{1}$ *for at least one* $A \in Y$.

Definition 6.3 (Consistency, inconsistency, validity, and contingence)**.** *A set of wffs* X *is consistent iff there exists a valuation* v *such that for each* $A \in X$, $v(A) = \mathbf{1}$*; otherwise* X *is inconsistent. A wff* B *is:*

1. *consistent iff the singleton set* $\{B\}$ *is consistent,*
2. *inconsistent iff the singleton set* $\{B\}$ *is inconsistent,*
3. *valid iff for each valuation* v, $v(B) = \mathbf{1}$,
4. *contingent iff* B *is neither inconsistent nor valid.*

[38] Where, however, a version of CPL with the equivalence connective, $\equiv$, was considered.

Remark 6.1. Consistent wffs construed in the above manner are often called satisfiable wffs. The category of contingent wffs comprises wffs which are satisfiable, but not valid.

Definition 6.4 (Logical equivalence). *Wffs A and B are logically equivalent iff $A \models B$ and $B \models A$.*

Sets of wffs, X and Y, are logically equivalent iff they have exactly the same models, where a model of a set of wffs is a valuation which makes true all the wffs in the set.

6.3 Strong Multiple-Conclusion Entailment

6.3.1 Definition and the Adequacy Issue

We use $\Vdash\!\prec$ as the symbol for strong mc-entailment, and we define the relation as follows:[39]

Definition 6.5 (Strong multiple-conclusion entailment; strong mc-entailment). *$X \Vdash\!\prec Y$ iff*

1. *$X \VDash Y$, and*
2. *for each $A \in X : X_{\ominus A} \not\Vdash Y$, and*
3. *for each $B \in Y : X \not\Vdash Y_{\ominus B}$.*

The consecutive clauses of the above definition express the following intuitions: the hypothetical truth of all the wffs in X warrants the existence of at least one true wff in Y, yet the warranty disappears as X decreases or Y decreases. In other words, X and Y are minimal sets under the warranty provided by the clause (1).

Here are simple examples:

$$\{p\} \Vdash\!\prec \{p\} \tag{6.1}$$

$$\{p, p \rightarrow q\} \Vdash\!\prec \{q\} \tag{6.2}$$

$$\{p \vee q, \neg p\} \Vdash\!\prec \{q\} \tag{6.3}$$

$$\emptyset \Vdash\!\prec \{p, \neg p\} \tag{6.4}$$

$$\{p, \neg p\} \Vdash\!\prec \emptyset \tag{6.5}$$

[39] Recall that $X_{\ominus A}$ abbreviates $X \setminus \{A\}$, and similarly for Y.

$$\emptyset \Vdash\!\prec \{p \vee \neg p\} \tag{6.6}$$

$$\{\neg p, \neg q, p \vee q\} \Vdash\!\prec \emptyset \tag{6.7}$$

$$\emptyset \Vdash\!\prec \{p, q, \neg(p \vee q)\} \tag{6.8}$$

$$\{p \vee q\} \Vdash\!\prec \{p, q\} \tag{6.9}$$

$$\{p \vee q\} \Vdash\!\prec \{p \wedge q, p \wedge \neg q, \neg p \wedge q\} \tag{6.10}$$

$$\{p \wedge q \rightarrow r, \neg r\} \Vdash\!\prec \{\neg p, \neg q\} \tag{6.11}$$

$$\{p \wedge (q \vee r)\} \Vdash\!\prec \{p \wedge q, p \wedge r\} \tag{6.12}$$

$$\{p \vee (q \vee r)\} \Vdash\!\prec \{p \vee q, p \vee r\} \tag{6.13}$$

$$\{\neg(p \wedge (q \wedge r))\} \Vdash\!\prec \{\neg p \vee \neg q, \neg p \vee \neg r\} \tag{6.14}$$

$$\{p \vee (q \vee r)\} \Vdash\!\prec \{(p \vee q) \wedge (p \vee r), q \vee r\} \tag{6.15}$$

Note that $\emptyset \Vdash\!\not\prec \emptyset$, as $\emptyset \not\Vdash \emptyset$.

Since the empty set has no proper subsets, and each proper subset of a non-empty set is included in a maximal proper subset of the set, it is clear that the following is true:

Corollary 6.1. *$X \Vdash\!\prec Y$ iff $X \Vdash Y$ and the following conditions hold:*

1. *there is no proper subset Z of X such that $Z \Vdash Y$,*
2. *there is no proper subset W of Y such that $X \Vdash W$.*

Due to the monotonicity of "standard" mc-entailment, $\Vdash$, we have:

Corollary 6.2. *If $X \Vdash\!\prec Y$, then:*

1. *$Z \Vdash\!\not\prec Y$, where Z is either a proper subset or a proper superset of X,*
2. *$X \Vdash\!\not\prec W$, where W is either a proper subset or a proper superset of Y.*

Thus strong mc-entailment, $\Vdash\!\prec$, is neither left-monotone nor right-monotone. The examples presented below witness this:

$$\{p, p \rightarrow q \vee r\} \Vdash\!\prec \{q, r\} \tag{6.16}$$

$$\{p, p \rightarrow q \vee r, \neg q\} \Vdash\!\not\prec \{q, r\} \tag{6.17}$$

$$\{p, p \to q \vee r\} \not\Vdash\!\prec \{q, r, q \vee r\} \tag{6.18}$$

Observe that the following are true:[40]

$$\{p, \neg p\} \not\Vdash\!\prec \{q\} \tag{6.19}$$

$$\{p\} \not\Vdash\!\prec \{p \vee \neg p\} \tag{6.20}$$

As for (6.19), $\{q\} \setminus \{q\} = \emptyset$, but we have $\{p, \neg p\} \Vdash\!= \emptyset$. In the case of (6.20) we have $\emptyset \Vdash\!= \{p \vee \neg p\}$.

Thus it is neither the case that any inconsistent set of wffs strongly mc-entails any set of wffs nor it is the case that a set which contains a valid wff is strongly mc-entailed by any set of wffs. Hence strong mc-entailment is free of the drawbacks (I') and (II') pointed out in section 6.1.2.

6.3.2 Strong Mc-entailment, Perfect Validity, and Tennant's Entailments

Corollary 6.1 yields that our concept of strong mc-entailment is akin to (but not identical with) the concept of perfectly valid sequent introduced by Neil Tennant in [48], p. 185.

Assume for a moment that sequents are simply pairs of sets of wffs. A proper subsequent of a sequent $X : Y$ is a sequent resulting from it by removing at least one wff from X or from Y. Tennant's definition of validity of a sequent $X : Y$ amounts to the presence of mc-entailment of Y from X. A sequent $X : Y$ is *perfectly valid* iff $X : Y$ is valid and no proper subsequent of $X : Y$ is valid. Thus, by Corollary 6.1, a sequent $X : Y$ is perfectly valid iff $X \Vdash\!\prec Y$ holds.[41]

However, perfect validity performs an auxiliary role in [48]. The central concept is that of sequent being an entailment. A sequent $X : Y$ is an *entailment* just in case $X : Y$ has a perfectly valid suprasequent. A sequent $Z : W$ is a suprasequent of the sequent $X : Y$ iff for some

[40] As for (6.19), $\{q\} \setminus \{q\} = \emptyset$, but we have $\{p, \neg p\} \Vdash\!= \emptyset$. In the case of (6.20) we have $\emptyset \Vdash\!= \{p \vee \neg p\}$.

[41] But if the concept of proper subsequent is to be understood differently (i.e. $X' : Y'$ is a proper subsequent of $X : Y$ just in case $X' \subsetneq X$ or $Y' \subsetneq Y$, one needs the condition:

$$X' \cup Y' \subseteq X \cup Y$$

in order to pass from perfect validity to strong mc-entailment. Tennant does not provide an explicit definition of the notion of proper subsequent used.

substitution s, $s(Z) = X$ and $s(W) = Y$. Tennant builds a sequent calculus which is sound and complete w.r.t. entailments construed in the above manner. A proof-theoretic account of perfectly valid sequents is also given by means of the so-called perfect proofs.

In this chapter we concentrate upon a semantic analysis of strong mc-entailment or, if you prefer, perfect validity. A proof-theoretic account of strong mc-entailment, different from that offered by Tennant for perfect validity, will be also provided in Chapter 10.

6.4 Basic Properties of Strong Mc-entailment

Let us first note:

Corollary 6.3. *Let A, B be logically equivalent wffs.*

1. *If $A \in X$ and $X \Vdash\!\prec Y$, then $X_{\ominus A} \cup \{B\} \Vdash\!\prec Y$.*
2. *If $A \in Y$ and $X \Vdash\!\prec Y$, then $X \Vdash\!\prec Y_{\ominus A} \cup \{B\}$.*

Thus logically equivalent wffs are replaceable in the context of strong mc-entailment. Needless to say, replaceability may fail for logical equivalence of sets of wffs. This is not surprising, as strong mc-entailment is a "hybrid" notion, defined in terms of semantic as well as set-theoretic clauses.[42]

Corollary 6.4. *$\{A\} \Vdash\!\prec \{A\}$ iff A is contingent.*

Proof. Clearly, $\{A\} \Vdash\!= \{A\}$, and $\{A\}_{\ominus A} = \emptyset$. On the other hand, A is not valid iff $\emptyset \Vdash\!\neq \{A\}$, and A is not inconsistent iff $\{A\} \Vdash\!\neq \emptyset$. □

However, the overlap/reflexivity condition is not satisfied in the case of non-singleton sets.

Corollary 6.5. *If X has at least two elements, then $X \Vdash\!\not\prec X$.*

Proof. Suppose otherwise. It follows that $X_{\ominus A} \Vdash\!\neq X$, where $A \in X$. But, as X has at least two elements, it holds that $X_{\ominus A} \cap X \neq \emptyset$ and hence $X_{\ominus A} \Vdash\!= X$. A contradiction. □

Let us now prove

Corollary 6.6. *If $X \Vdash\!\prec Y$ and X is inconsistent, then $Y = \emptyset$.*

[42] Such a solution has obvious vices, but also some virtues; see sections 7.2.3 and 8.1 below.

Proof. Let $X \Vert\!\prec Y$. Thus $X \Vert\!\models Y$. Assume that X is inconsistent. Suppose that $Y \neq \emptyset$. Thus $\emptyset$ is a proper subset of Y. However, $X \Vert\!\models \emptyset$ (since X is inconsistent) and hence $X \Vert\!\not\prec Y$ due to Corollary 6.1. So $Y = \emptyset$. □

Thus an inconsistent set strongly mc-entails, if any, only the empty set. If any, since there are inconsistent sets that do not strongly mc-entail even the empty set. For instance, the set $\{p \wedge \neg p, p\}$ does not strongly entail the empty set because we still have $\{p \wedge \neg p\} \Vert\!\models \emptyset$. As we will see, only minimally inconsistent sets strongly mc-entail the empty set.

Remark 6.2. There exist strongly mc-entailed inconsistent sets of wffs. Examples (5.26) and (6.8) presented above support this claim. Here are examples which do not involve the empty set:

$$\{p\} \Vert\!\prec \{p \wedge q, p \wedge \neg q\} \tag{6.21}$$

$$\{\neg(p \wedge q), p \vee q\} \Vert\!\prec \{p \wedge \neg q, \neg p \wedge q\} \tag{6.22}$$

6.4.1 Contingent, Valid, and Inconsistent Wffs

Interestingly enough, strong mc-entailment between non-empty sets of wffs involves sets which comprise contingent wffs only. The following holds:

Theorem 6.1 (Contingency). *Let $X \Vert\!\prec Y$. If $X \neq \emptyset$ and $Y \neq \emptyset$, then each wff in $X \cup Y$ is contingent.*

Proof. Assume that $X \Vert\!\prec Y$, where X and Y are non-empty sets.

Suppose that X contains a valid wff, say, A. It follows that $X_{\ominus A} \Vert\!\models Y$ and therefore $X \Vert\!\not\prec Y$. Now suppose that X contains an inconsistent wff. Hence X is an inconsistent set. But $Y \neq \emptyset$. Thus, by Corollary 6.6, $X \Vert\!\not\prec Y$, which contradicts the assumption. Therefore X contains contingent wffs only.

Suppose that a valid wff, say, A, belongs to Y. By assumption, $X \neq \emptyset$, so $\emptyset$ is a proper subset of X. Suppose that $Y = \{A\}$. Clearly, $\emptyset \Vert\!\models \{A\}$ due to the validity of A. Hence $X \Vert\!\not\prec \{A\}$. Now suppose that $Y \neq \{A\}$. As $Y \neq \emptyset$, it follows that $\{A\}$ is a proper subset of Y which, however, is mc-entailed by X since A is valid. Thus $X \Vert\!\not\prec Y$. Therefore no valid wff belongs to Y.

Finally, suppose that an inconsistent wff, B, belongs to Y. In this case $X \Vert\!\models Y$ yields $X \Vert\!\models Y_{\ominus B}$. As Y is, by assumption, non-empty,

$Y_{\ominus B}$ is a proper subset of Y. It follows that $X \Vdash\!\not\prec Y$. We arrive at a contradiction. Thus no wff in Y is inconsistent. Therefore $X \cup Y$ contains contingent wffs only. $\square$

What if either X or Y is empty? The answer is provided by:

Corollary 6.7.

1. *If $\emptyset \Vdash\!\prec Y$, then either Y is a singleton set containing a valid wff, or Y is a non-singleton set comprising only contingent wffs.*
2. *If $X \Vdash\!\prec \emptyset$, then either X is a singleton set containing an inconsistent wff, or X is a non-singleton set comprising only contingent wffs.*

Proof. If $\emptyset \Vdash\!\prec Y$, then $Y \neq \emptyset$. Assume that Y is a singleton set, $\{C\}$. Since $\emptyset \Vdash\!\prec \{C\}$ presupposes $\emptyset \Vdash\!= \{C\}$, it follows that C is a valid wff. Assume that Y is a non-singleton set. Suppose that Y contains a non-contingent wff, say, B. If B is valid, then $\emptyset \Vdash\!= \{B\}$ and hence $\emptyset \Vdash\!\not\prec Y$. The situation is analogous when B is an inconsistent wff; in this case we would have $\emptyset \Vdash\!= Y_{\ominus B}$.

If $X \Vdash\!\prec \emptyset$, then $X \neq \emptyset$. Assume that X is a singleton set, $\{C\}$. Thus $\{C\} \Vdash\!= \emptyset$ and hence C is an inconsistent wff. Assume that X is a non-singleton set. Suppose that X contains a non-contingent wff, say, A. Clearly, $X_{\ominus A}$ is a proper subset of X and so is $\{A\}$. Assume that A is valid. Thus $X_{\ominus A} \Vdash\!= \emptyset$ and hence $X \Vdash\!\prec \emptyset$ does not hold. Now assume that A is inconsistent. Thus $\{A\} \Vdash\!= \emptyset$ and hence, again, $X \Vdash\!\prec \emptyset$ is not the case. Therefore each wff in X is contingent provided that X is a non-singleton set. $\square$

As for strong mc-entailment, non-contingent wffs come into play in two exceptional situations only.

Theorem 6.2. *Let $X \Vdash\!\prec Y$.*

1. *If C is valid, then: $C \in X \cup Y$ iff $X = \emptyset$ and $Y = \{C\}$.*
2. *If C is inconsistent, then: $C \in X \cup Y$ iff $X = \{C\}$ and $Y = \emptyset$.*

Proof. Let C be a valid wff. Assume that $X \Vdash\!\prec Y$ and $C \in X \cup Y$.

Suppose that $C \in X$. Hence $X \neq \emptyset$ and $X_{\ominus C}$ is a proper subset of X. If C is valid, then whatever is mc-entailed by X is also mc-entailed by $X_{\ominus C}$. So $X \Vdash\!\not\prec Y$. We arrive at a contradiction. Therefore $C \notin X$ and thus $C \in Y$.

Suppose that $X \neq \emptyset$. Thus $\emptyset$ is a proper subset of X. Since C is valid and $C \in Y$, we have $\emptyset \,\|\!\models Y$. It follows that $X \,\|\!\not\prec Y$, contrary to the assumption. Therefore $X = \emptyset$. As $C \in Y$, it follows that Y does not comprise contingent wffs only. Hence $Y = \{C\}$ due to Corollary 6.7.

Needless to say, if $Y = \{C\}$, then $C \in X \cup Y$.

The proof of (*2*) goes along similar lines. □

According to Theorem 6.2, valid wffs can occur as elements of strongly mc-entailed sets, but these sets are always singleton sets which, moreover, are strongly mc-entailed only by the empty set. Similarly, if an inconsistent wff belongs to a strongly mc-entailing set, it is the only element of this set and the respective strongly mc-entailed set is empty. Moreover, valid wffs never occur in strongly mc-entailing sets, and inconsistent wffs never occur in strongly mc-entailed sets.

6.4.2 Strict Finiteness and Variable Sharing

We are dealing here with CPL, in which mc-entailment has the following properties:

(lf) *If* $X \,\|\!\models Y$, *then* $X_1 \,\|\!\models Y$ *for some finite subset* X_1 *of* X.

(rf) *If* $X \,\|\!\models Y$, *then* $X \,\|\!\models Y_1$ *for some finite subset* Y_1 *of* Y.

As for CPL (and other logics in which mc-entailment fulfils the above conditions), strong mc-entailment is strictly finitistic in the sense explained by:

Theorem 6.3 (Strict finiteness). *If* $X \,\|\!\prec Y$, *then* X *and* Y *are finite sets.*

Proof. Let $X \,\|\!\prec Y$. Suppose that X is an infinite set. By Corollary 6.1, it follows that there is no finite subset of X which mc-entails Y. Hence $X \,\|\!\not\models Y$ due to condition (lf). But $X \,\|\!\prec Y$ yields $X \,\|\!\models Y$. So X is a finite set. Now suppose that Y is an infinite set. Hence, by Corollary 6.1, no finite subset of Y is mc-entailed by X. Thus $X \,\|\!\not\models Y$ due to condition (rf). It follows that $X \,\|\!\prec Y$ does not hold, contrary to the assumption. So Y is a finite set as well. □

Our next theorem is strongly dependent on the fact that we consider here propositional formulas.

Notation. By $\mathtt{Var}(A)$ we designate the set of all the propositional variables that occur in a wff A. $\mathtt{Var}(X)$ designates the set of all the propositional variables that occur in the wffs which belong to a set of wffs X.

Theorem 6.4 (Variable sharing). *Let $X \Vdash\!\prec Y$. If X and Y are non-empty sets, then $\mathtt{Var}(X) \cap \mathtt{Var}(Y) \neq \emptyset$.*

Proof. Let $X \Vdash\!\prec Y$, where $X \neq \emptyset$ and $Y \neq \emptyset$.

If $X \neq \emptyset$, then, by Corollary 6.2, $\emptyset \not\Vdash Y$. By assumption, $Y \neq \emptyset$. So there exists a valuation, say, v^*, such that $v^*(B) = \mathbf{0}$ for any $B \in Y$. By Corollary 6.6, X is consistent. Hence there exists a valuation v such that $v(A) = \mathbf{1}$ for every $A \in X$.

Suppose that $\mathtt{Var}(X) \cap \mathtt{Var}(Y) = \emptyset$. Let v^+ be a valuation such that: (a) $v^+(p_i) = v^*(p_i)$ if $p_i \in \mathtt{Var}(Y)$, (b) otherwise $v^+(p_i) = v(p_i)$. As $\mathtt{Var}(X) \cap \mathtt{Var}(Y) = \emptyset$, we have $v^+(A) = \mathbf{1}$ for every $A \in X$. On the other hand, $v^+(B) = \mathbf{0}$ for each $B \in Y$. Hence $X \not\Vdash Y$ and therefore $X \Vdash\!\not\prec Y$. We arrive at a contradiction. □

So when strong mc-entailment between X and Y holds, the wffs in X share propositional variable(s) with the wffs in Y. However, Theorem 6.4 cannot be strengthened to the effect that $\mathtt{Var}(Y) \subseteq \mathtt{Var}(X)$ would be the case. Similarly, $\mathtt{Var}(X) \subseteq \mathtt{Var}(Y)$ does not generally hold.[43]

6.5 Partial Reduction to Minimally Inconsistent Sets

As long as a logic operating with the classical negation is concerned, there exist simple links between strong mc-entailment and minimally inconsistent sets:[44]

Definition 6.6 (Minimally inconsistent set; MI-set). *A set of wffs X is minimally inconsistent iff X is inconsistent, but each proper subset of X is consistent.*

[43] For instance, we have $\{p \vee q\} \Vdash\!\prec \{p, r \to q\}$ as well as $\{p \wedge q\} \Vdash\!\prec \{p\}$.

[44] The concept of minimally inconsistent set has found natural applications is many areas, from philosophy of science (cf., e.g., [26]) to theoretical computer science, AI, and logic (see, e.g., [32], [6], [37]). Minimally inconsistent sets are also called *minimal unsatisfiable (sub)sets* or *unsatisfiable cores*.

For brevity, we will be referring to minimally inconsistent sets as to MI-sets.

Note that $\emptyset$ is not a MI-set. Singleton MI-sets have inconsistent wffs as the (only) elements. Here are examples of non-singleton MI-sets:

$$\{p, \neg p\} \tag{6.23}$$

$$\{p \vee q, \neg p, \neg q\} \tag{6.24}$$

$$\{p \to q, p, \neg q\} \tag{6.25}$$

$$\{p \to q \vee r, p, \neg q, \neg r\} \tag{6.26}$$

$$\{p \to q, q \to r, \neg(p \to r)\} \tag{6.27}$$

Clearly, the following holds:

Corollary 6.8. *X is a* MI*-set iff X is inconsistent and for each $A \in X$, the set $X_{\ominus A}$ is consistent.*

Remark 6.3. As for CPL, any MI-set is finite. This is due to the fact that the following *compactness claim* holds for CPL:

(♣) *for each set of wffs Z: the set Z is consistent iff each finite subset of Z is consistent.*

However, there are logics for which the analogues of (♣) do not hold and thus finiteness is not a property of MI-sets in general.[45]

Notation. For brevity, we put:

$$\neg Y =_{df} \{\neg A : A \in Y\}$$

In the case of CPL, strong mc-entailment and MI-sets are linked in the following way:

Theorem 6.5 (Partial reduction to MI-sets for $\Vert\!\!\prec$)**.** *$X \Vert\!\!\prec Y$ iff $X \cap \neg Y = \emptyset$ and $X, \neg Y$ is a* MI*-set.*

[45] For example, in a logic that validates the ω-rule, a set of the form

$$\{\exists x \mathtt{P}x\} \cup \{\neg \mathtt{Pa} : \mathtt{a} \in \mathbb{T}\}$$

where $\mathtt{P}$ is a predicate and $\mathbb{T}$ is a (countably infinite) set of all closed terms of the language, is an infinite MI-set.

Proof. ($\Rightarrow$) Let $X \Vdash\!\prec Y$. Suppose that $X \cap \neg Y \neq \emptyset$. Let $A \in X \cap \neg Y$. Thus $A = \neg B$ for some $B \in Y$. Let $Y^* = Y_{\ominus B}$. From $X \Vdash\!\prec Y$ we get $X \Vdash\!= Y^*, B$. Therefore $X, \neg B \Vdash\!= Y^*$, that is, $X, A \Vdash\!= Y^*$. But $X, A = X$, since $A \in X$. Hence X mc-entails the proper subset Y^* of Y. It follows that $X \Vdash\!\not\prec Y$. We arrive at a contradiction. Therefore $X \cap \neg Y = \emptyset$.

If $X \Vdash\!\prec Y$, then $X \Vdash\!= Y$ and thus the set $X, \neg Y$ is inconsistent. Let us designate the set $X, \neg Y$ by Z.

If $A \in Z$, then $A \in X$ or $A \in \neg Y$.

Assume that $A \in X$. By the clause 2 of Definition 6.5, $X_{\ominus A} \Vdash\!\not= Y$ and thus the set $X_{\ominus A}, \neg Y$ is consistent, that is, $Z_{\ominus A}$ is consistent.

Now assume that $A \in \neg Y$. Hence $A = \neg B$ for some $B \in Y$. By the clause 3 of Definition 6.5, $X \Vdash\!\not= Y_{\ominus B}$. Thus the set $X, \neg(Y_{\ominus B})$ is consistent. Yet, $X, \neg(Y_{\ominus B}) = Z_{\ominus A}$. Hence the set $Z_{\ominus A}$ is consistent.

By Corollary 6.8, $X, \neg Y$ is thus a MI-set.

($\Leftarrow$) Assume that $X \cap \neg Y = \emptyset$ and $X, \neg Y$ is a MI-set. From the latter it follows that $X \Vdash\!= Y$.

Again, let $Z = X, \neg Y$.

Suppose that $X_{\ominus A} \Vdash\!= Y$ for some $A \in X$. Then the set $X_{\ominus A}, \neg Y$ is inconsistent. Yet, since $X \cap \neg Y = \emptyset$, the set $X_{\ominus A}, \neg Y$ is a proper subset of Z. Thus Z is not a MI-set. A contradiction.

Now suppose that $X \Vdash\!= Y_{\ominus B}$ for some $B \in Y$. Let us designate $Y_{\ominus B}$ by Y^*. As $X \Vdash\!= Y^*$ holds, the set $X, \neg Y^*$ is inconsistent. But $X \cap \neg Y = \emptyset$, so $\neg B$ does not belong to X. Hence the set $X, \neg Y^*$ is a proper subset of Z. Thus Z is not a MI-set. A contradiction again.

Therefore $X \Vdash\!\prec Y$. □

Theorem 6.5 yields:

Corollary 6.9.

1. $X \Vdash\!\prec \emptyset$ *iff* X *is a* MI*-set.*
2. $\emptyset \Vdash\!\prec Y$ *iff* $\neg Y$ *is a* MI*-set.*

Remark 6.4. As the second part of the proof of Theorem 6.5 shows, one can get $X \Vdash\!\prec Y$ from the fact that $X, \neg Y$ is a MI-set *on the condition* that $X \cap \neg Y = \emptyset$ holds. This condition is a necessary one. For example, let $X = \{p \vee q, \neg p, \neg q\}$ and $Y = \{p, q\}$. Then $\neg Y = \{\neg p, \neg q\}$ and hence $X, \neg Y = X$. As X is a MI-set, so is $X, \neg Y$. However, $X \Vdash\!\not\prec Y$, since $\{p \vee q\} \Vdash\!= \{p, q\}$. On the other hand, $X \cap \neg Y = \{\neg p, \neg q\} \neq \emptyset$.

There exist MI-sets which do not contain wffs beginning with negation, i.e. wffs of the form $\neg B$. Here are simple examples:

$$\{p \to q, p \wedge \neg q\}$$

$$\{p, p \to q, p \to \neg q\}$$

It may seem that such MI are "useless" in showing that strong mc-entailment holds. But this is wrong. The corollary below explains why.

Corollary 6.10. *If X, Y is a* MI*-set and $X \cap Y = \emptyset$, then $X \Vdash\!\prec \neg Y$.*

Proof. Clearly, if X, Y is a MI-set, then $X, \neg(\neg Y)$ is a MI-set. Suppose that $X \cap \neg(\neg Y) \neq \emptyset$. So there exists $A \in X$ such that $A = \neg\neg B$ for some $B \in Y$, and $A \in \neg(\neg Y)$. As X, Y is a MI-set and $X \cap Y = \emptyset$, we have $B \notin X$ and thus the set $X, Y_{\ominus B}$ is consistent. Hence $X, (\neg(\neg Y))_{\ominus\neg\neg B}$ is a consistent set as well. But $X, (\neg(\neg Y))_{\ominus\neg\neg B} = X, \neg(\neg Y)$, since $A = \neg\neg B$ and $A \in X$. It follows that $X, \neg(\neg Y)$ is not a MI-set. We arrive at a contradiction. Thus $X \cap \neg(\neg Y) = \emptyset$. As $X, \neg(\neg Y)$ is a MI-set, by Theorem 6.5 we get $X \Vdash\!\prec \neg Y$. □

6.6 Independence and Deduction

Observe that if X strongly mc-entails Y, then neither X nor Y contains syntactically distinct wffs which are logically equivalent, i.e. entail each other. The reason is that a MI-set never includes logically equivalent wffs. We can also prove more:

Theorem 6.6 (Independence)**.** *Let $X \Vdash\!\prec Y$, and let A, B be syntactically distinct wffs.*

1. *If $A, B \in X$ and $Y \neq \emptyset$, then $A \not\models B$ and $A \not\models \neg B$.*
2. *If $A, B \in Y$, then $A \not\models B$, and $\neg A \not\models B$ provided that $\{A, B\} \neq Y$.*

Proof. If $X \Vdash\!\prec Y$, then, by Theorem 6.5, $X, \neg Y$ is a MI-set and $X \cap \neg Y = \emptyset$.

Let $A, B \in X$. Thus the set $X_{\ominus B}, \neg Y$ is consistent and, due to the fact that $X, \neg Y$ is inconsistent, $X_{\ominus B}, \neg Y \models \neg B$. But $A \in X_{\ominus B}$. Therefore $X_{\ominus B}, \neg Y \models A$. Hence $A \not\models B$.

As $B \in X$, we have $X \models B$. Suppose that $A \models \neg B$. Since $A \in X$, it follows that $X \models \neg B$. Thus X is an inconsistent set and, as $Y \neq \emptyset$, we get $X \not\Vdash\!\prec Y$.

Let $A, B \in Y$. It follows that $\neg A, \neg B \in \neg Y$. By Theorem 6.5, $X, \neg Y$ is a MI-set and hence an inconsistent set. Thus $X, \neg(Y_{\ominus A}) \models A$. However, the set $X, \neg(Y_{\ominus A})$, as a proper subset of the MI-set in question, is consistent. On the other hand, '$\neg B$' $\in \neg(Y_{\ominus A})$. It follows that $X, \neg(Y_{\ominus A}) \models \neg B$. Therefore $A \not\models B$.

Assume that $\{A, B\} \neq Y$. Suppose that $\neg A \models B$. It follows that $\emptyset \models \neg A \rightarrow B$ and hence $\emptyset \Vert\!\models \{A, B\}$. As $\{A, B\} \neq Y$, we get $X \Vert\!\not\prec Y$. □

Notation. For conciseness, let us introduce the following notational convention:

$$\lceil A \rightarrow W \rceil =_{df} \begin{cases} \{\neg A\} \ \textit{if} \ \ W = \emptyset, \\ \{A \rightarrow B : B \in W\} \ \textit{if} \ \ W \neq \emptyset. \end{cases}$$

One can easily show that the following holds:

Corollary 6.11. *$Z, A \Vert\!\models W$ iff $Z \Vert\!\models \lceil A \rightarrow W \rceil$.*

As a consequence we get:

Theorem 6.7 (Deduction for strong mc-entailment)**.** *Let $A \notin X$. If $X, A \Vert\!\prec Y$, then $X \Vert\!\prec \lceil A \rightarrow Y \rceil$.*

Proof. Assume that $X, A \Vert\!\prec Y$. If $X, A \Vert\!\models Y$, then, by Corollary 6.11, $X \Vert\!\models \lceil A \rightarrow Y \rceil$. Let $B \in X$. Since, by assumption, $A \notin X$, it follows that $A \neq B$. Thus $X_{\ominus B}, A$ is a proper subset of X, A. As $X, A \Vert\!\prec Y$ holds, we have $X_{\ominus B}, A \Vert\!\not\models Y$. Hence, by Corollary 6.11 again, $X_{\ominus B} \Vert\!\not\models \lceil A \rightarrow Y \rceil$. Let $C \in Y$. Thus $X, A \Vert\!\not\models Y_{\ominus C}$. Therefore, by Corollary 6.11, $X \Vert\!\not\models \lceil A \rightarrow Y_{\ominus C} \rceil$. Hence $X \Vert\!\prec \lceil A \rightarrow Y \rceil$. □

Note that the converse of Theorem 6.7 is not true. For example, $\emptyset \Vert\!\prec \{p \rightarrow q, p \rightarrow \neg q\}$ holds, but $\{p\} \Vert\!\prec \{q, \neg q\}$ is not the case. However, the following is true:

Corollary 6.12. *If $X \Vert\!\prec \lceil A \rightarrow Y \rceil$ and $X \not\models \neg A$ as well as $X \Vert\!\not\models Y$, then $X, A \Vert\!\prec Y$.*

Proof. Suppose that $Y = \emptyset$. Hence $X \Vert\!\prec \{\neg A\}$. Thus $X \models \neg A$. But, by assumption, $X \not\models \neg A$. So $Y \neq \emptyset$.

If $X \Vert\!\prec \lceil A \rightarrow Y \rceil$, then, by Definition 6.5 and Corollary 6.11, $X, A \Vert\!\models Y$ and $X_{\ominus B} \cup \{A\} \Vert\!\not\models Y$ for any $B \in X$. By assumption, $X \Vert\!\not\models Y$. It follows that for every $C \in X, A$ we have $X, A \setminus \{C\} \Vert\!\not\models Y$. Now suppose that $X, A \Vert\!\models Y_{\ominus D}$ is the case for some $D \in Y$. There are two possibilities:

(a) $Y_{\ominus D} = \emptyset$ and (b) $Y_{\ominus D} \neq \emptyset$. Assume that (a) holds. It follows that the set X, A is inconsistent. But, by assumption, $X \not\models \neg A$ and hence the set X, A is consistent. So (a) does not hold. It follows that Y is not a singleton set. Assume that (b) is the case. Therefore, by Corollary 6.11, $X \Vdash\!= \lceil A \to Y_{\ominus D} \rceil$. As $\lceil A \to Y_{\ominus D} \rceil$ is a proper subset of $\lceil A \to Y \rceil$, it follows that $X \not\Vdash\!\prec \lceil A \to Y \rceil$. So we arrive at a contradiction again. Hence $X, A \not\Vdash\!= Y_{\ominus D}$ for every $D \in Y$. As all the clauses of Definition 6.5 are fulfilled w.r.t. X, A and Y, we conclude that $X, A \Vdash\!\prec Y$ holds. □

As the proof of Corollary 6.12 shows, the assumption "$X \not\models \neg A$" is dispensable when Y is neither a singleton set nor the empty set.

Corollary 6.13. *Let Y be a finite and at least two-element set of wffs. If $X \Vdash\!\prec \lceil A \to Y \rceil$ and $X \not\Vdash\!= Y$, then $X, A \Vdash\!\prec Y$.*

Finally, observe that $\Vdash\!\prec$ is not closed under uniform substitution. A simple example illustrates this. Clearly, $\{p\} \Vdash\!\prec \{p\}$ is the case. But $\{p \wedge \neg p\} \Vdash\!\prec \{p \wedge \neg p\}$ does not hold (cf. Corollary 6.6). Needless to say, $p \wedge \neg p$ results from p by substitution.

Chapter 7

Strong Single-Conclusion Entailment

7.1 Definition and the Adequacy Issue

Sc-entailment traditionally construed can be identified with mc-entailment of a singleton set. Similarly, it seems natural to define strong sc-entailment as strong mc-entailment of a singleton set.

We use $|\!\!<$ as the symbol for strong sc-entailment.

Definition 7.1 (Strong single-conclusion entailment; strong sc-entailment)**.** *$X |\!\!< B$ iff $X \|\!\!< \{B\}$.*

For brevity, we will write $A |\!\!< B$ instead of $\{A\} |\!\!< B$.

As an immediate consequence of Definition 7.1 and Theorem 6.5 one gets:

Theorem 7.1 (Partial reduction to MI-sets for $|\!\!<$)**.** *$X |\!\!< B$ iff '$\neg B$' $\notin X$ and $X, \neg B$ is a* MI*-set.*

Note that the transition from right to left requires '$\neg B$' $\notin X$ to hold. For example, although

$$\{p, p \to \neg q, \neg\neg q\} \cup \{\neg\neg q\} \tag{7.1}$$

is a MI-set, $\{p, p \to \neg q, \neg\neg q\} |\!\!< \neg q$ does not hold, since '$\neg\neg q$' $\in \{p, p \to \neg q, \neg\neg q\}$.

The following is true:

Corollary 7.1. *$X \mathrel{|\!\!<} B$ iff*

1. *$X \models B$ and*
2. *for each proper subset Z of X: $Z \not\models B$, and*
3. *X is consistent.*

Proof. Clearly, $X \models B$ holds iff $X \Vdash \{B\}$ is the case.

Clause (*2*) holds due to Corollary 6.1. On the other hand, clause (*2*) yields that there is no $A \in X$ such that $X_{\ominus A} \Vdash \{B\}$.

Since $\{B\} \setminus \{B\} = \emptyset$, clause (*3*) of Definition 6.5 and clause (*3*) of the above corollary are equivalent for $Y = \{B\}$. □

Strong sc-entailment is not monotone. As a matter of fact, it is "antimonotone" in a sense explained by:

Corollary 7.2. *If $X \mathrel{|\!\!<} B$ and $X \subseteq Y$, where $Y \neq X$, then $Y \not\mathrel{|\!\!<} B$.*

Proof. By Definition 7.1 and Corollary 6.2. □

As we pointed out in section 6.1.1, the monotonicity of entailment contravenes, in a sense, the semantic entrenchment idea, since it allows semantically irrelevant wffs to occur among premises. In the case of strong sc-entailment, however, the difficulty is solved in a radical way: a strongly sc-entailing set is "minimal" with regard to the transmission of truth and, since no proper superset of a set X that strongly sc-entails a wff B strongly sc-entails B as well, adding an "irrelevant" wff to X results in the lack of strong sc-entailment of B from X enriched in this way.

By the clause (*2*) of Corollary 7.1, each proper subset of a strongly sc-entailing set is consistent. Strong sc- and mc-entailment do not differ in this respect. As we have seen, however, there exist strongly mc-entailing sets which are inconsistent (each of them strongly mc-entails only the empty set, however). According to the clause (*3*) of Corollary 7.1, this never happens in the case of strong sc-entailment. Anyway, strong sc-entailment is free of the drawback (I) pointed out in section 6.1.1. Let us add: free, again, in a radical way, since inconsistent sets do not strongly sc-entail any wffs. As an immediate consequence of Corollary 6.6 one gets:

Corollary 7.3. *No wff is strongly sc-entailed by an inconsistent set of wffs.*

Thus no inconsistent wff belongs to a sc-entailing set, and a singleton set which comprises an inconsistent wff does not strongly sc-entail any wff. In particular, neither $A \wedge \neg A \models\!\!< A$ nor $\{A, \neg A\} \models\!\!< A$ holds, regardless of what A is. Similarly, there is no B such that $A \wedge \neg A \models\!\!< B$ or $\{A, \neg A\} \models\!\!< B$.

Observe that the following holds as well:

Corollary 7.4. *There is no set of wffs that strongly sc-entails an inconsistent wff.*

Proof. By Definition 7.1 and Theorem 6.2. □

Thus inconsistencies are outside the realm of strong sc-entailment: no inconsistent set belongs to the domain of $\models\!\!<$ and no inconsistent wff belongs to the range of the relation. No doubt, a paraconsistent logician will dislike strong sc-entailment.

The case of validities is slightly more complicated. By Theorem 6.2 we get:

Corollary 7.5. *If $X \models\!\!< B$, then no wff in X is valid.*

Corollary 7.6. *If B is valid and $X \models\!\!< B$, then $X = \emptyset$.*

One can prove that valid wffs are exactly these wffs which are strongly sc-entailed only by the empty set.

Corollary 7.7. *A wff B is valid iff $\emptyset \models\!\!< B$ and $X \not\models\!\!< B$ for any $X \neq \emptyset$.*

Proof. Let B be a valid wff. Thus $\{\neg B\}$ is a MI-set, and hence, by Corollary 6.9, $\emptyset \models\!\!< B$. Thus $X \not\models\!\!< B$ for any $X \neq \emptyset$. On the other hand, if $\emptyset \models\!\!< B$, then $\emptyset \models B$ and hence B is valid. □

As for valid wffs, Corollary 7.6 yields that the difference between strong sc-entailment and sc-entailment simpliciter lies in the fact that valid wffs are strongly sc-entailed *only* by the empty set. Thus, in particular, valid wffs are not strongly sc-entailed by sets of valid wffs. Moreover, a valid wff is not sc-entailed by any set of wffs to which a valid wff belongs to.

7.2 Some Properties of Strong Sc-entailment

Since strong sc-entailment is defined in terms of strong mc-entailment, one can easily derive the following corollaries from the corresponding results presented in sections 6.4.1, 6.4.2, and 6.6.

Corollary 7.8. *Let A, B be logically equivalent wffs.*

1. *If $A \in X$ and $X \mid\!\!\prec C$, then $X_{\ominus A} \cup \{B\} \mid\!\!\prec C$.*
2. *If $X \mid\!\!\prec A$, then $X \mid\!\!\prec B$.*

Corollary 7.9 (Contingency for $\mid\!\!\prec$)**.** *If $X \mid\!\!\prec B$ and $X \neq \emptyset$, then each wff in $X \cup \{B\}$ is contingent.*

Corollary 7.10 (Strict finiteness of $\mid\!\!\prec$)**.** *If $X \mid\!\!\prec B$, then X is a finite set.*

Corollary 7.11 (Variable sharing for $\mid\!\!\prec$)**.** *If $X \mid\!\!\prec B$ and $X \neq \emptyset$, then $\mathtt{Var}(X) \cap \mathtt{Var}(B) \neq \emptyset$.*

Corollary 7.12 (Independence for $\mid\!\!\prec$)**.** *Let $X \mid\!\!\prec B$. If A, C are syntactically distinct wffs that belong to X, then $A \not\models C$ and $A \not\models \neg C$.*

Corollary 7.13 (Deduction for $\mid\!\!\prec$)**.** *Let $A \notin X$. If $X, A \mid\!\!\prec B$, then $X \mid\!\!\prec A \to B$.*

The converse of Corollary 7.13 is not true. For instance, $\emptyset \mid\!\!\prec p \wedge \neg p \to q$ holds, but $p \wedge \neg p \mid\!\!\prec q$ does not hold. Yet, there are cases in which $X \mid\!\!\prec A \to B$ yields $X, A \mid\!\!\prec B$. Corollary 6.12 implies:

Corollary 7.14. *If $X \mid\!\!\prec A \to B$, and $X \not\models \neg A$ as well as $X \not\models B$, then $X, A \mid\!\!\prec B$.*

Thus we get:

Corollary 7.15. *Let $A \notin X$, and $X \not\models \neg A$ as well as $X \not\models B$. Then $X, A \mid\!\!\prec B$ iff $X \mid\!\!\prec A \to B$.*

Proof. By corollaries 7.13 and 7.14. □

Strong sc-entailment is, in a sense, closed under detachment.

Corollary 7.16 (Detachment for $\mid\!\!\prec$)**.** *If $X \mid\!\!\prec A \to B$ and $X \mid\!\!\prec A$, then $X \mid\!\!\prec B$.*

Proof. Either $X \mid\!\!\prec A \to A$ or $X \mid\!\!\prec B$ warrants the consistency of X, and together they yield that $X \models B$ holds.

Assume that $X \neq \emptyset$. Let C be an arbitrary but fixed element of X. From $X \mid\!\!\prec A \to B$ we get $X_{\ominus C} \not\models A \to B$. It follows that $X_{\ominus C} \not\models B$. Thus $X \mid\!\!\prec B$.

Now assume that $X = \emptyset$. In this case B is a valid wff. Therefore $\emptyset \mid\!\!\prec B$ due to Corollary 7.7, that is, $X \mid\!\!\prec B$. □

Observe that one can also prove that $X \mid\!\!\prec A \to B$ and $X \models A$ yield $X \mid\!\!\prec B$.

7.2.1 Strong Sc-entailment from Singleton Sets

Strong sc-entailment from single wffs (more precisely, from singleton sets of wffs) has some properties which strong sc-entailment from non-singleton sets lack.

Corollary 7.17. *The following are equivalent:*

1. $A \mathrel{|\!\!<} B$,
2. $A \models B$ *and* A, B *are contingent wffs.*

Proof. The implication from (*1*) to (*2*) is due to Definition 7.1 and Corollary 7.9. As for the passage from (*2*) to (*1*), it suffices to observe that the contingency of A warrants the consistency of $\{A\}$, while the contingency of B guarantees that $\emptyset \Vert\!\!\models \{B\}$ does not hold. □

One cannot generalize Corollary 7.17 to non-singleton sets. The contingency of all the wffs belonging to a (non-empty) non-singleton set of wffs X warrants neither the consistency of X itself nor the lack of entailment of B from proper subset(s) of X.

Coming back to sc-entailment from single wffs. The lack of strong sc-entailment in the presence of standard sc-entailment tells us more about the wffs involved than Corollary 7.17 does.

Corollary 7.18. *If* $A \models B$*, but* $A \not\mathrel{|\!\!<} B$*, then* A *is inconsistent or* B *is valid.*

Proof. If $A \models B$ and $A \not\mathrel{|\!\!<} B$, then $\{A\}$ is an inconsistent set or $\emptyset \models B$. So A is inconsistent or B is valid. □

When X is a non-empty set having more that one element, the lack of $X \mathrel{|\!\!<} B$ in the presence of $X \models B$ implies that X is inconsistent or B is entailed by some proper subset of X.

Finally, let us notice the following:

Corollary 7.19. *If* $A \mathrel{|\!\!<} B$ *and* $B \mathrel{|\!\!<} C$*, then* $A \mathrel{|\!\!<} C$.

Proof. Certainly, $A \models B$ and $B \models C$ yields $A \models C$. By Corollary 7.17, $A \mathrel{|\!\!<} B$ warrants the contingency of A, while $B \mathrel{|\!\!<} C$ yields the contingency of C. So $A \mathrel{|\!\!<} C$ due to Corollary 7.17. □

Observe that when X has more than one element, the passage from $X \mathrel{|\!\!<} B$ and $B \mathrel{|\!\!<} C$ to $X \mathrel{|\!\!<} C$ requires an additional condition to be met, namely it must be ensured that for each $D \in X$, the set $X_{\ominus D}$ does not entail C.

7.2.2 Mutuality

As for CPL, mc-entailment of a non-empty finite set reduces to sc-entailment of a disjunction of all the elements of the set, i.e. if Y is a finite set and $Y \neq \emptyset$, then $X \Vert\!\models Y$ iff $X \models \bigvee Y$. But strong mc-entailment and strong sc-entailment are not linked in this way. For instance, we have:

$$p \mathrel{|\!\prec} p \vee q \tag{7.2}$$

but we *do not* have:[46]

$$p \mathrel{\|\!\prec} \{p, q\} \tag{7.3}$$

Strong mc- and sc-entailments are mutually linked in a quite different way, as the following theorem shows.

Theorem 7.2 (Mutuality)**.**

1. *If* $X \mathrel{\|\!\prec} Y, B$, *where* $B \notin Y$, *then* $X, \neg Y \mathrel{|\!\prec} B$.
2. *If* $X, \neg Y \mathrel{|\!\prec} B$ *and* $X \cap \neg Y = \emptyset$, *then* $X \mathrel{\|\!\prec} Y, B$.

Proof. ($\Rightarrow$). If $X \mathrel{\|\!\prec} Y, B$, then, by Theorem 6.5, $X \cup (\neg Y \cup \{\neg B\})$ is a MI-set and $X \cap (\neg Y \cup \{\neg B\}) = \emptyset$. It follows that $(X \cup \neg Y) \cup \{\neg B\}$ is a MI-set and '$\neg B$' $\notin X$. By assumption, $B \notin Y$. So '$\neg B$' $\notin \neg Y$. Hence '$\neg B$' $\notin X, \neg Y$. Thus $X, \neg Y \mathrel{|\!\prec} B$ by Theorem 7.1.
($\Leftarrow$). If $X, \neg Y \mathrel{|\!\prec} B$, then, by Theorem 7.1, $(X \cup \neg Y) \cup \{\neg B\}$ is a MI-set and '$\neg B$' $\notin X \cup \neg Y$. Suppose that $X \cap (\neg Y \cup \{\neg B\}) \neq \emptyset$. As '$\neg B$' $\notin X \cup \neg Y$, it follows that $(X \cap \neg Y) \neq \emptyset$. On the other hand, by assumption $(X \cap \neg Y) = \emptyset$. Thus $X \cap (\neg Y \cup \{\neg B\}) = \emptyset$. Therefore $X \mathrel{\|\!\prec} Y, B$ due to Theorem 6.5. □

7.2.3 Conjunction versus Set of Conjuncts

As for the standard sc-entailment based on Classical Logic, there is no scope difference between being entailed by a finite set of wffs and being entailed by a conjunction of all the wffs of this set. Although conjunction, $\wedge$, is semantically construed here in the classical manner (cf. section 6.2), it is worth to note that strong sc-entailment from a conjunction of wffs and strong sc-entailment from a set of all its conjuncts only overlap, but not coincide. Clearly, the following is true:

Corollary 7.20. *Let* $X \neq \emptyset$. *If* $X \mathrel{|\!\prec} B$, *then* $\bigwedge X \mathrel{|\!\prec} B$.

[46] (7.3) does not hold because $\{p\} \Vert\!\models (\{p, q\} \setminus \{q\})$.

For example, $\{p, q\} \mathrel{|\!\!<} p \wedge q$ is the case and thus $p \wedge q \mathrel{|\!\!<} p \wedge q$ holds as well. Yet, the converse of Corollary 7.20 is not true. For instance, $p \wedge q \mathrel{|\!\!<} p$ holds, while $\{p, q\} \mathrel{|\!\!<} p$ does not hold.[47] At first sight this looks untenable. However, the phenomenon can be explained as follows. Information carried by $\bigwedge X \mathrel{|\!\!<} B$ and $X \mathrel{|\!\!<} B$ differ when X is not a singleton set. In both cases transmission of truth as well as consistency of the set X are ensured. The claim of $\bigwedge X \mathrel{|\!\!<} B$ is: although B need not be true, the (hypothetical) truth of all the wffs in X is sufficient for B be true. Note that $\bigwedge X \mathrel{|\!\!<} B$ does not exclude that the transmission of truth effect takes place w.r.t. some proper subset or some proper superset of X. (As for $p \wedge q \mathrel{|\!\!<} p$, there is a proper subset of $\{p, q\}$, namely $\{p\}$, which ensures the transmission.) The claim of $X \mathrel{|\!\!<} B$ is stronger: this is just the (hypothetical) truth of all the wffs in X that warrants the (hypothetical) truth of B. "Just" means here: "one needs neither more nor less than the truth of *all* the wffs in X for B be true."

Observe that one can pass from $\bigwedge X \mathrel{|\!\!<} B$ to $X \mathrel{|\!\!<} B$ on the condition:

($\heartsuit$) *for each* $A \in X : \bigwedge(X_{\ominus A}) \not\models B$

which, however, does not hold universally.

Remark 7.1. Although strong sc-entailment is "antimonotone" (cf. Corollary 7.2), the following fact is worth some attention:

Corollary 7.21. *Let $X \neq \emptyset$. If $X \mathrel{|\!\!<} B$ and Y is a consistent proper superset of X, then $\bigwedge Y \mathrel{|\!\!<} B$.*

Proof. By Corollary 7.6, if $X \neq \emptyset$ and $X \mathrel{|\!\!<} B$, then B is not valid. So $\emptyset \not\models B$. On the other hand, $\emptyset$ is the only proper subset of $\{\bigwedge Y\}$. Clearly, if $X \models B$, then $Y \models B$ and hence $\{\bigwedge Y\} \models B$. If Y is consistent, so is $\{\bigwedge Y\}$. Therefore $\bigwedge Y \mathrel{|\!\!<} B$. □

Thus a wff strongly sc-entailed by a non-empty set of wffs X is also strongly sc-entailed by (the singleton set comprising) a conjunction of all the wffs of a consistent extension Y of X. Note, however, that, according to what has been said above, $\bigwedge Y \mathrel{|\!\!<} B$ carries less information than $X \mathrel{|\!\!<} B$. Moreover, $X \mathrel{|\!\!<} B$ suppresses $Y \mathrel{|\!\!<} B$.

[47] By the way, these examples provide a nice illustration of the lack of transitivity of strong sc-entailment.

7.2.4 Transposition and Abduction

Let us end this chapter by pointing out some interesting property of strong sc-entailment.

Theorem 7.3. *Let $A \notin X$ and '$\neg B$' $\notin X$.*

1. *If $X, A \mid\!\!< B$, then $X, \neg B \mid\!\!< \neg A$.*
2. *If $X, \neg B \mid\!\!< \neg A$, then $X, A \mid\!\!< B$.*

Proof. Assume that $X, A \mid\!\!< B$. Thus $X, A \models B$ and hence $X, \neg B \models \neg A$.

Suppose that $X, \neg B$ is an inconsistent set. It follows that $X \models B$ holds and therefore $X, A \not\mid\!\!< B$. Thus the set $X, \neg B$ is consistent.

Now suppose that $(X \cup \{\neg B\}) \setminus \{C\} \models \neg A$ for some $C \in X \cup \{\neg B\}$.

Assume that $C = \neg B$. Hence $X \models \neg A$ and thus the set X, A is inconsistent. Hence, by Corollary 7.1, $X, A \not\mid\!\!< B$. A contradiction. Now assume that $C \neq \neg B$. It follows that $C \in X$. Thus there exists a proper subset, Y, of X such that $Y, \neg B \models \neg A$. If follows that $Y, A \models B$ and therefore, as $A \notin X$ and thus $A \notin Y$, the set Y, A entails B. But Y, A is a proper subset of X, A. Hence we again get $X, A \not\mid\!\!< B$. But by assumption it holds that $X, A \mid\!\!< B$.

Therefore $X, \neg B \mid\!\!< \neg A$ by Corollary 7.1.

Assume that $X, \neg B \mid\!\!< \neg A$. Thus $X, \neg B \models \neg A$ and therefore $X, A \models B$.

Suppose that X, A is an inconsistent set. Thus $X, \neg\neg A$ is inconsistent as well. It follows that $X \models \neg A$ is the case and, since '$\neg B$' $\notin X$, it is not the case that $X, \neg B \mid\!\!< \neg A$. A contradiction.

Suppose that for some $C \in (X \cup \{A\})$ it holds that $(X \cup \{A\}) \setminus \{C\} \models B$. Assume that $C = A$. Thus (as $A \notin X$) we have $X \models B$. It follows that the set $X, \neg B$ is inconsistent and thus, by Corollary 7.1, $X, \neg B \not\mid\!\!< \neg A$.

Finally, suppose that $C \neq A$. Hence there exists $D \in X$ such that $(X \setminus \{D\}) \cup \{A\} \models B$. Therefore $(X \setminus \{D\}) \cup \{\neg B\} \models \neg A$. But '$\neg B$' $\notin X$. Thus $(X \setminus \{D\}) \cup \{\neg B\}$ is a proper subset of $X, \neg B$ which entails $\neg A$. It follows that $X, \neg B \mid\!\!< \neg A$ does not hold. We arrive at a contradiction again.

Therefore, by Corollary 7.1, $X, A \mid\!\!< B$. □

According to Theorem 7.3, strong sc-entailment of a wff B from a set of wffs X enriched with a wff A yields strong sc-entailment of the negation of A from X extended with the negation of B, and strong sc-entailment of the negation of A from X enriched with the negation of

B yields strong sc-entailment of B from X extended with A. So if you have a set of wffs X, a formula B which is not entailed by X and whose negation does not belong to X, and you aim at extending X to the effect that B becomes strongly sc-entailed by the extended set (that is, you want to fill the "deductive gap" between X and B in a non-tricky way)[48], you look for formulas whose negations are strongly sc-entailed by the set $X \cup \{\neg B\}$.

Example 7.22. $p \to q$ does not entail q. But we have:

$$\{p \to q, \neg q\} \models\!\!\!< \neg p \tag{7.4}$$

By applying Theorem 7.3 to (7.4), we get:

$$\{p \to q, p\} \models\!\!\!< q \tag{7.5}$$

Example 7.23. $p \to q$ does not entail $p \to r$. However, the following holds:

$$\{p \to q, \neg(p \to r)\} \models\!\!\!< \neg(q \to r) \tag{7.6}$$

Thus, by Theorem 7.3:

$$\{p \to q, q \to r\} \models\!\!\!< p \to r \tag{7.7}$$

Example 7.24. The set $X = \{p \to q, q \to r\}$ does not entail r. However, we have:

$$\{p \to q, q \to r, \neg r\} \models\!\!\!< \neg p \tag{7.8}$$

and hence:

$$\{p \to q, q \to r, p\} \models\!\!\!< r \tag{7.9}$$

Thus the "deductive gap" between X and r can be filled with p. But it is obvious that the gap can also be filled with q. However, since we do not have:

$$\{p \to q, q \to r, \neg r\} \models\!\!\!< \neg q \tag{7.10}$$

Theorem 7.3 is of no use in indicating this solution. On the other hand, enriching X with p ensures strong sc-entailment of r from the enriched set, while extending X by q does not produce this effect.

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48 There are many tricky ways of filling a deductive gap between X and B: one can extend X to an inconsistent set, or one can simply add B or a formula which entails B to X. Neither of these moves, however, would ensure that B becomes strongly sc-entailed by the extended set.

Remark 7.2. The so-called abductive problem is sometimes described as follows: given the set of formulas X and a formula B such that X does not entail B, find a formula A such that X and A together entail B.[49] Usually, constraints are imposed on "good" solutions to the problem. For example, it is required that (a) $X \cup \{B\}$ must be a consistent set, and that (b) B is entailed neither by only X nor by only A. Further criteria are imposed as well (see, e.g., [49]). Now observe that once we require the relation between $X \cup \{A\}$ and B to be strong sc-entailment rather than classical entailment, criteria (a) and (b) are met automatically. Moreover, $X \cup \{A\}$ is a "minimal" consistent set such that the hypothetical truth of all the wffs in it warrants the truth of B.

[49] A warning is in order: this short statement does not exhaust what abduction is or is conceived to be; cf. [49].

Chapter 8

Strong Entailments: Comparisons

8.1 Strong versus Classical

The basic properties of strong entailments differ from those of their classical counterparts. However, one can show that whatever is reachable by classical entailments from consistent sets of premises, is also attainable by strong entailments from some finite subsets of these sets. To put it briefly: no classical consequence of a consistent set is lost.

Notice that it holds that (we present a proof of this well-known fact only to keep this chapter self-contained):

Lemma 8.1. *Each inconsistent set of wffs has a subset being a* MI*-set.*

Proof. Let X be an inconsistent set of wffs. By compactness of CPL, X has an inconsistent finite subset, say, X'. Clearly, $X' \neq \emptyset$. Consider the family of all inconsistent subsets of X'. Let us designate it by Ψ. Since X' is inconsistent, $\Psi \neq \emptyset$. As X' is non-empty and finite, there is a natural number, say, k, where $k \geqslant 1$, such that no set in Ψ has less than k elements. Let Y be an element of Ψ which comprises exactly k wffs. Obviously, no proper subset of Y belongs to Ψ. Therefore each proper subset of Y is consistent. It follows that Y is a MI-set included in X. □

Let us now prove:

Theorem 8.1 (Simulation of $\Vert\!\models$). *If $X \Vert\!\models Y$ and X is consistent, then there exist a finite subset X_1 of X and a finite non-empty subset Y_1 of Y such that $X_1 \Vert\!\prec Y_1$.*

Proof. If $X \Vdash Y$, then the set $X, \neg Y$ is inconsistent and thus, by Lemma 8.1, has subset(s) being MI-sets. Let Z be a MI-set such that $Z \subseteq X, \neg Y$. Since, by assumption, X is consistent, $Z \nsubseteq X$. We put:

$$X_1 =_{df} X \cap Z$$

$$W =_{df} Z \setminus X_1$$

Clearly, $W \subseteq \neg Y$. Moreover, $W \neq \emptyset$, and $Z = X_1, W$ as well as $X_1 \cap W = \emptyset$. Consider the set Y_1 defined by:

$$Y_1 =_{df} \{C : \neg C \in W\}.$$

We have $W = \neg Y_1$ and hence $Z = X_1, \neg Y_1$. It follows that $Y_1 \subseteq Y$ and $X_1 \cap \neg Y_1 = \emptyset$. Since Z is a MI-set and $Z = X_1, \neg Y_1$ as well as $X_1 \cap \neg Y_1 = \emptyset$, by Theorem 6.5 we conclude that $X_1 \Vdash\!\prec Y_1$ holds. As each MI-set is finite, X_1 and Y_1 are finite subsets of X and Y, respectively. Finally, $Y_1 \neq \emptyset$ since $W \neq \emptyset$. □

As a consequence of Definition 7.1 and Theorem 8.1 we get:

Theorem 8.2 (Simulation of $\models$)**.** *If $X \models B$ and X is consistent, then there exists a finite subset Z of X such that $Z \models\!\prec B$.*

Proof. Recall that $X \models B$ iff $X \Vdash \{B\}$, and $Z \models\!\prec B$ iff $Z \Vdash\!\prec \{B\}$. Since we have already proven Theorem 8.1, it suffices to observe that the only non-empty subset of the singleton set $\{B\}$ is $\{B\}$ itself. □

The intuitive content of Theorem 8.2 is this: CPL sc-entailment from a given, finite or infinite, consistent set of wffs boils down to strong sc-entailment from a finite subset of the set. Theorem 8.1 presents an analogous result for mc-entailment.

Remark 8.1. Let X and Y be different, yet logically equivalent consistent sets of wffs. The set of wffs classically sc-entailed by X coincides with the set of wffs classically entailed by Y. However, this need not be the case for strong sc-entailment. Yet, Theorem 8.2 yields that the set of wffs attainable by strong sc-entailment from some finite subset of X equals the set of wffs which are obtainable by strong sc-entailment from some finite subset of Y, and equals the set comprising all the wffs classically sc-entailed by X or by Y. Needless to say, the respective subsets of X and of Y may differ.

8.2 Strong versus Relevant

As it is well-known, when the sum of two consistent sets of CPL-wffs, X and Y, is inconsistent, then $\mathtt{Var}(X) \cap \mathtt{Var}(Y) \neq \emptyset$ (cf., e.g. ,[17], p. 375). It follows that classical sc-entailment from consistent sets of premises to conclusions which are not valid wffs exhibits the variable sharing property. It is worth to note that the same holds true for strong sc-entailment. Theorem 8.2 together with corollaries 7.11 and 7.7 almost immediately yield:

Corollary 8.1. *Let X be a non-empty, consistent set of wffs. If $X \models B$ and B is not a valid wff, then there exists a finite, non-empty subset Z of X such that $Z \Vdash\!\!< B$ and $\mathtt{Var}(Z) \cap \mathtt{Var}(B) \neq \emptyset$.*

Variable sharing is often regarded as an indicator (or even a precondition) of relevance in the context of semantic consequence. As such, it is usually invoked in relevant logics. So the question arises: what is the relation between strong sc-entailment and accounts of entailment proposed in relevant logics? Since there exist many systems of relevance logic, an exhaustive answer would have required a separate paper. Let me restrict here to a few remarks only.

As for CPL, valid wffs falling under the schema:

$$A \to B \tag{8.1}$$

license sc-entailment of B from A. For the lack of a better idea, let us call them *classical implicational laws* or briefly CIL's.[50]

Recall that although all classically valid wffs are strongly sc-entailed by the empty set, the transition from $\emptyset \Vdash\!\!< A \to B$ to $A \Vdash\!\!< B$ is not always legitimate (cf. Corollary 7.15). Corollary 7.17 yields, in turn, that a CIL does not license strong sc-entailment just in case its antecedent or consequent is not contingent.

The first observation is: there exist CIL's which are both rejected in some relevant logics[51] and do not license strong sc-entailment. Examples are shown in Table 8.1.

Second, there exist CIL's which are rejected in some relevant logic(s), but license strong sc-entailment. Examples are given in Table 8.2.

[50] One should not confuse CIL's with laws of the implicational fragment of CPL. Both A and B may involve any connective, implication included. What is important is that implication is the main connective of (the wff which expresses) a CIL.

[51] That is, at least one of well-known relevant logics rejects the corresponding law; there is no space for details. For relevant logics see, e.g., [33] and [38].

$p \wedge \neg p \to q$	$p \wedge \neg p \not\Vdash q$
$\neg(p \to p) \to q$	$\neg(p \to p) \not\Vdash q$
$p \to (q \to q)$	$p \not\Vdash q \to q$
$p \to (p \to p)$	$p \not\Vdash p \to p$
$p \to p \vee \neg p$	$p \not\Vdash p \vee \neg p$
$p \to q \vee \neg q$	$p \not\Vdash q \vee \neg q$
$(p \to p) \to (q \to q)$	$p \to p \not\Vdash q \to q$
$(p \to q) \to (p \to p)$	$p \to q \not\Vdash p \to p$

Table 8.1: *Examples of* CIL*'s rejected in some relevant logics (left column) which do not license strong sc-entailment (as depicted in the right column).*

$p \to (q \to p)$	$p \Vdash (q \to p)$
$p \to (\neg p \to q)$	$p \Vdash (\neg p \to q)$
$p \to ((p \to q) \to q)$	$p \Vdash (p \to q) \to q$
$((p \to q) \to p) \to p$	$(p \to q) \to p \Vdash p$
$(p \to (q \to r)) \to (q \to (p \to r))$	$p \to (q \to r) \Vdash q \to (p \to r)$
$p \wedge q \to (p \to q) \wedge (q \to p)$	$p \wedge q \Vdash (p \to q) \wedge (q \to p)$
$p \wedge (\neg p \vee q) \to q$	$p \wedge (\neg p \vee q) \Vdash q$

Table 8.2: *Examples of* CIL*'s rejected in some relevant logics (left column) which, however, license strong sc-entailment (as depicted in the right column).*

Third, it happens that a CIL which is accepted in a relevant logic does not license strong sc-entailment. The "mingle" formulas, i.e. wffs of the form $A \to (A \to A)$, provide simple examples here.

8.3 Strong versus Connexive

Connexive logics are usually characterized as systems validating the following theses:[52]

$$\neg(\neg A \to A) \tag{8.2}$$

[52] For connexive logics see, e.g., [35]. Theses (8.2) and (8.3) are attributed to Aristotle, while theses (8.4) and (8.5) are ascribed to Boethius.

$$\neg(A \to \neg A) \tag{8.3}$$

$$(A \to B) \to \neg(A \to \neg B) \tag{8.4}$$

$$(A \to \neg B) \to \neg(A \to B) \tag{8.5}$$

As strong sc-entailment from the empty set is restricted to classically valid wffs only (cf. Corollary 7.7) and some wffs of the forms (8.2) – (8.5) are not classically valid, it is not the case that all the wffs falling under the schemata (8.2) – (8.5) are strongly sc-entailed by the empty set.[53] It is worth to note, however, that the following are true:

Corollary 8.2. *For any wff A:*

1. $\neg A \not\models\!\!< A$,
2. $A \not\models\!\!< \neg A$.

Proof. Suppose that $\neg A \models\!\!< A$ for some wff A. Thus $\neg A \models A$. On the other hand, by Theorem 6.1 it follows that both $\neg A$ and A are contingent wffs. Hence $\neg A \not\models A$.

We reason analogously in the case of (*2*). □

Thus a negation of a wff never strongly sc-entails the wff itself, and a wff never strongly sc-entails its negation. Corollary 8.2 seems to express an idea akin to that which lies behind having (8.2) and (8.3) as theses.

Corollary 8.3. *For any wffs A, B:*

1. *if $A \models\!\!< B$, then $A \not\models\!\!< \neg B$,*
2. *if $A \models\!\!< \neg B$, then $A \not\models\!\!< B$,*

Proof. Assume that $A \models\!\!< B$. It follows that A is a consistent wff and $A \models B$. Therefore there exists a valuation v such that $v(A) = \mathbf{1}$, $v(B) = \mathbf{1}$ and hence $v(\neg B) = \mathbf{0}$. Thus $A \not\models \neg B$. It follows that $A \not\models\!\!< \neg B$.

We reason similarly in the case of (*2*). □

A due comment on Corollary 8.3 is analogous to that on Corollary 8.2.

[53] However, some of them are classically valid and thus *are* strongly sc-entailed by the empty set. For instance, we have $\emptyset \models\!\!< \neg(\neg(p \wedge \neg p) \to p \wedge \neg p)$, $\emptyset \models\!\!< \neg((p \to p) \to \neg(p \to p))$, or $\emptyset \models\!\!< (p \vee \neg p \to p \wedge \neg p) \to \neg(p \vee \neg p \to \neg(p \wedge \neg p))$.

8.4 The First-Order Case

In chapters 6 and 7 we have dealt with the classical propositional case. So a natural question arises: what, if anything, will change when we move to the first-order level and consider strong entailments based on First-Order Logic (FOL)?

As it is well-known, sc-entailment in FOL can be defined either in terms of satisfaction or in terms of truth, and similarly for mc-entailment. However, truth of a wff in a FOL-model equals satisfaction of the wff *under all assignments* of values to individual variables, where the values belong to the universe of the model. Therefore the respective concepts of entailment do not coincide when sentential functions, that is, wffs in which free variables occur, enter the picture, although they coincide on FOL-sentences (i.e. wffs with no free variables). Similarly, inconsistency can be defined either as unsatisfiability or as the lack of a FOL-model which makes true all the wffs in question. These are not the same thing if sentential functions are allowed.[54]

When one wants to move from the propositional level to the first-order one, three possibilities emerge.

The simplest solution is to assume that strong entailments, as well as the other semantic notions employed, are defined for sentences only. The concept of truth under a CPL-valuation is to be replaced with the concept of truth in a FOL-model. Then the results concerning CPL "translate" into the respective results concerning the "sentential part" of FOL. Of course, this does not pertain to results which rely on the assumption that the wffs considered are propositional, in particular to Theorem 6.4. Needless to say, an analogous remark applies to the other options presented below.

The second option is to allow for sentential functions and to replace "true under a CPL-valuation" with "satisfied in a FOL-model under an assignment of values to individual variables." In such a case inconsistency would mean unsatisfiability. There is, however, a price to be paid. While sc-entailment defined in terms of satisfaction ensures the transmission of truth, mc-entailment defined by means of satisfaction (i.e. roughly, by the clause: "for every assignment ι: if all the wffs in X are satisfied under ι, then at least one wff in Y is satisfied under ι") does not warrant the existence of a *true* wff in Y when all the wffs in X are true. This lack of

[54] For instance, the set $\{\mathtt{P}(x), \neg\forall x \mathtt{P}(x)\}$, where $\mathtt{P}$ is a one-place predicate, is satisfiable, but there is no FOL-model which makes its elements simultaneously true.

warranty shows up in the case of mc-entailed sets containing sentential functions. As a consequence, the intuitive meaning of the concept of strong mc-entailment changes.

As for the third option, one allows for sentential functions and replaces "true under a CPL-valuation" with "true in a FOL-model." Now consistency of a set of wffs would mean the existence of a FOL-model which makes all the wffs true. Contingent wffs are these which are true in some, but not all FOL-models. However, sc-entailment of A from X amounts to inconsistency of the set comprising X and *the negation of the universal closure* of A. Similarly, mc-entailment between X and Y holds iff the set $X, \neg\overline{Y}$ is inconsistent, where $\overline{Y}$ is the set of universal closures of elements of Y. So a "translation" of results concerning CPL should be performed with caution. In particular, whenever consistency/inconsistency of propositional formulas of the form $\neg A$ or sets of such formulas have been considered, first-order wffs of the form $\neg\overline{A}$, where $\overline{A}$ is the universal closure of A, should be used. For example, the FOL counterparts of theorems 6.5 and 7.1 now are:

$$X \,\|\!\!\prec Y \quad \textit{iff} \quad X \cap \neg\overline{Y} = \emptyset \textit{ and } X, \neg\overline{Y} \textit{ is a } \mathsf{MI}\textit{-set.}$$

$$X \,|\!\!\prec B \quad \textit{iff} \quad `\neg\overline{B}\,' \notin X \textit{ and } X, \neg\overline{B} \textit{ is a } \mathsf{MI}\textit{-set.}$$

Another example is this. What we have called "deduction theorems" for strong entailments (cf. theorems 6.7 and 7.13), relied upon Corollary 6.11. However, its counterpart does not hold for FOL when entailments are defined in terms of truth. Instead, we have:

$$Z, \overline{A} \,\|\!\!\models W \textit{ iff } Z \,\|\!\!\models \ulcorner \overline{A} \to W \urcorner.$$

As a consequence, in order to get counterparts of theorems 6.7 and 7.13 one has to replace A with $\overline{A}$. An analogous remark pertains to corollaries 6.12, 7.14, and 7.15. As for Theorem 7.3, both A and B are to be replaced with their universal closures.

8.5 Strong Entailments in Non-Classical Logics

In our analysis of strong entailments we have concentrated upon Classical Logic. A natural next step is to turn to non-classical logics. Which of the results presented above would remain valid if we defined strong entailments in terms of entailments based on a non-classical logic? No doubt, this is an interesting question. Yet, it deserves a separate paper or even a series of papers. So let me only comment on the relation between the concepts of strong entailments and the concept of minimally inconsistent set. Theorems 6.5 and 7.1 (as well as their counterparts for FOL) show how these concepts are interconnected for Classical Logic. However, analogues of theorems 6.5 and 7.1 fail in some non-classical logics. Negationless logics provide trivial examples here, but there are others. For instance, in Intuitionistic Logic (INT) the following:

$$\{\neg\neg p, \neg p\} \models_{\mathsf{INT}} \bot$$

$$\{\neg\neg p\} \not\models_{\mathsf{INT}} \bot$$

$$\{\neg p\} \not\models_{\mathsf{INT}} \bot$$

hold and thus $\{\neg\neg p, \neg p\}$ can be regarded as a MI-set. Needless to say, '$\neg p$' $\notin \{\neg\neg p\}$. On the other hand, we have:

$$\neg\neg p \not\models_{\mathsf{INT}} p$$

and hence, assuming that strong sc-entailment presupposes sc-entailment, $\neg\neg p$ and p are not linked with strong sc-entailment. It follows that the "intuitionistic" analogues of theorems 7.1 and 6.5 do not hold.

Chapter 9

Deep Contraction

The concepts of strong entailment have found applications in argument analysis and in the area of belief revision (cf. [65]). Since argument analysis is not in the scope of this book, in this chapter I concentrate upon the latter issue, extending (in section 9.3 below) the material already published in [65].

9.1 Intuitions

Let us imagine that we are working with a consistent non-empty set of CPL-wffs X (for instance, representing a database or a belief base) and that a contingent CPL-wff B has been derived from X. Assume that the derivation mechanism used preserves CPL-entailment. Now suppose that we have strong, though independent from X, reasons to believe that $\neg B$ rather than B is the case. As long as we stick to Classical Logic, extending X with $\neg B$ is not a good move. An option is to switch to some non-monotonic logic and its consequence operation. As we have shown, strong sc-entailment is not monotone. But no extension of X produces $\neg B$ as a strongly sc-entailed consequence of X. This is due to:

Corollary 9.1. *If $X \models\!\!\!< B$, then there is no proper superset Z of X such that $Z \models\!\!\!< \neg B$.*

Proof. Let $X \models\!\!\!< B$. Suppose that $Z \models\!\!\!< \neg B$, where $X \subseteq Z$ and $X \neq Z$. If $X \models\!\!\!< B$, then $X \models B$ and hence $Z \models B$. If $Z \models\!\!\!< \neg B$, then $Z \models \neg B$. Therefore Z is inconsistent and thus, by Corollary 7.1, $Z \not\models\!\!\!< \neg B$. □

A rational move is to contract X first, and in a way that prevents the appearance of B as a conclusion of any legitimate (i.e. preserving classical entailment) derivation from the contracted set. How to achieve this? One can examine the derivation of B from X that has actually been performed, identify the elements of X used as premises, and then contract X by removing from it at least one wff which was used as a premise in the performed derivation. This, however, will not do: it is possible that B is classically entailed by many subsets of X, including some that do not contain the just removed wff(s), and thus B can still be legitimately derived from the set contracted in the above manner. Examining all possible legitimate derivations of B from X constitutes a difficult if not a hopeless task. However, a solution is suggested by the content of Theorem 8.2. By and large, it suffices to consider all the finite subsets of X that strongly sc-entail B, and to remove from X exactly one element of every such subset. A contracted set obtained in this way does not CPL-entail the wff B and therefore no legitimate derivation leads from the set to B. Or, to put it differently, X has been deeply contracted w.r.t. B, since it is ensured that B is not entailed by the result of the contraction.

Remark 9.1. The way of proceeding proposed above is akin to (but not identical with) the well-known idea of consistency restoring by calculating a minimal hitting set of the family of all minimally inconsistent subsets of an inconsistent set in order to eliminate elements of the hitting set from the inconsistent set in question.[55] The general idea goes back to [36] and gave rise to some related constructions.[56] However, contraction of the kind we are interested in this chapter does not aim at consistency restoring, but at making a legitimate deduction of B from the resultant set impossible. These are interconnected, but yet different issues.

In order to model deep contraction we will make use of some concepts already introduced in section 1.7.1 of Chapter 1. Among them, the most useful is the concept of a ch-set of a family of sets of wffs Φ (cf. Definition 1.11 in Chapter 1). A $\mathtt{ch}(\Phi)$-set is a set comprising exactly one *representative* of each non-empty set belonging to Φ. Intuitively, a $\mathtt{ch}(\Phi)$-set can be construed as a "choice set": one chooses from each set

[55] A set X is a hitting set of a family of sets $\mathbb{F}$ iff $X \cap Y \neq \emptyset$ for each $Y \in \mathbb{F}$. A hitting set of $\mathbb{F}$ is minimal if no proper subset of it is a hitting set of $\mathbb{F}$. Hitting sets are also called choice sets. For hitting/choice sets see, e.g., [47], pp. 335– 338.

[56] Cf. e.g., [6].

in Φ a wff which performs the function of representative of the set. In can be proven (cf. Proposition 1.10) that for each family of sets of wffs, Φ, there exists at least one $\mathtt{ch}(\Phi)$-set.

9.2 Deep Contraction Modelled

Let us now come back to the contraction issue. The following holds.

Theorem 9.1 (Deep contraction). *Let X be a consistent non-empty set of wffs, and let B be a non-valid wff such that $X \models B$. Let*

$$\Phi = \{W \subseteq X : W \mid\!\!\ltimes B\},$$

and let Z be a $\mathtt{ch}(\Phi)$-set. Then $(X \setminus Z) \not\models B$.

Proof. By Theorem 8.2, the family Φ is non-empty. If B is non-valid, $\emptyset \notin \Phi$. Thus $X' \neq \emptyset$ for each $X' \in \Phi$, and hence $Z \neq \emptyset$.

The set $X \setminus Z$ is consistent, since X is, by assumption, consistent.

Suppose that $(X \setminus Z) \models B$. It follows that $(X \setminus Z) \neq \emptyset$ (as B is not valid) and, by Theorem 8.2, that $Y \mid\!\!\ltimes B$ for some finite subset Y of $X \setminus Z$. Moreover, $Y \neq \emptyset$; otherwise B would have been valid. But the only subsets of X that strongly sc-entail B are the sets in Φ. Hence $Y = X^{\circ}$ for some element, X°, of Φ. But $(X' \cap Z) \neq \emptyset$ for each $X' \in \Phi$. Hence $(X^{\circ} \cap Z) \neq \emptyset$. On the other hand, $(Y \cap Z) = \emptyset$ due to the fact that Y is a subset of $X \setminus Z$. It follows that $Y \neq X^{\circ}$. We arrive at a contradiction. Therefore $(X \setminus Z) \not\models B$. □

Let us stress that Theorem 9.1 speaks about *any* $\mathtt{ch}$-set of the family of subsets of X which strongly sc-entail B. There are usually many such sets. Each of them may be subtracted from X in order to arrive at a subset of X that does not (classically) entail B. In other words, "deep contraction" can be successfully performed in many ways and its outcome depends on the $\mathtt{ch}$-set chosen.

A simple example may be of help.

Example 9.25. Let $X = \{p \vee q \rightarrow r, p, q\}$ and $B = r$. The relevant family of subsets of X each of which sc-entails r comprises: $\{p \vee q \rightarrow r, p\}$ and $\{p \vee q \rightarrow r, q\}$; let us designate it by Φ. The $\mathtt{ch}(\Phi)$-sets are:

$$\{p, q\} \tag{9.1}$$

$$\{p \vee q \rightarrow r\} \tag{9.2}$$

The result of deep contraction of X w.r.t. r, depending on the $\mathtt{ch}(\Phi)$-set used, is:

$$\{p \vee q \rightarrow r\} \tag{9.3}$$

or

$$\{p, q\} \tag{9.4}$$

Which $\mathtt{ch}(\Phi)$-set is to be used depends on epistemic factors. By the way, the example presented above shows that deep contraction does not amount to subtracting a minimal choice set of the family of all MI-sets in question.

Belief revision theories view contraction as an operation which is supposed to achieve its goal(s) in an "economical" manner: the loss should be kept to a minimum. This means many things, depending on an account advocated. As for deep contraction, the "minimalization of loss" issue is only of a secondary importance. As for the multiplicity of possible outcomes, and their dependence on factors different from the set subjected to be contracted and the wff w.r.t. which the operation is performed, deep contraction does not differ from other contraction operations characterized in belief revision theories. Note, however, that deep contraction has a kind of computational flavour. In order to perform it one needs a $\mathtt{ch}$-set of the family of subsets of X which strongly sc-entail B, and this requires that the family has to be "calculated" first. Given the content of Theorem 6.5, this, in turn, can be achieved by identifying all the minimally inconsistent subsets of an inconsistent set of some kind. More specifically, all minimally inconsistent subsets of $X \cup \{\neg B\}$ such that $\neg B$ belongs to each of them have to be identified first. Then the family:

$$\{Y \subseteq X : Y \cup \{\neg B\} \textit{ is a } \mathsf{MI}\textit{-set and } `\neg B' \notin Y\} \tag{9.5}$$

which comprises all the subsets of X which sc-entail B, is taken into consideration and its $\mathtt{ch}$-sets are calculated. Finally, one subtracts a $\mathtt{ch}$-set of the family (9.5) from X and arrives at a subset of X which does not entail B anymore.

Algorithms for identifying all minimally inconsistent subsets of an inconsistent set are already known in the literature.[57]

Remark 9.2. A set of wffs X supposed to be contracted w.r.t. B may be either finite or infinite. In the latter case it can happen that the family of subsets of X that sc-entail B is countably or even uncountably

[57] See, e.g., [32], and [6].

infinite. It follows that the relevant ch-sets may be infinite. However, we are dealing here with Classical Logic, in which entailment is compact: everything entailed by an infinite set of wffs is also entailed by some finite subset(s) of the set. One can easily prove:

Corollary 9.2. *Let X be an infinite consistent set of wffs, and let B be a non-valid wff such that $X \models B$. If Y is a finite subset of X such that $Y \models B$, and Z is a* $\mathtt{ch}(\Phi)$*-set, where* $\Phi = \{W \subseteq Y : W \mid\!\!\!< B\}$*, then* $X \cap (Y \setminus Z) \not\models B$.

Proof. Suppose otherwise. Then $(Y \setminus Z) \models B$, contrary to Theorem 9.1. □

Thus when entailment is compact, an infinite set X can also be contracted to the effect that B is not entailed by the resultant set, without relying on infinite ch-set(s) that correspond(s), in the way described above, to the whole X. It suffices to use a ch-set which corresponds to a finite subset Y of X that classically entails B. Needless to say, the resultant set $X \cap (Y \setminus Z)$ will be finite.

9.3 A Procedure of Identifying All Strongly Entailing Subsets of a Finite Set of Premises

In this section a procedure by means of which one can identify all strongly entailing subsets of a consistent and finite set of wffs that entails a given wff, is sketched. According to what has been said above, their identification is a prerequisite for a deep contraction w.r.t. the entailed wff.

Let X be a finite consistent set of wffs such that $X \models B$. Our aim is to find all the finite subsets of X that strongly sc-entail B.

If B is a valid wff, then $\emptyset$ is the only finite subset of X that strongly sc-entails B. So the problem is solved in a trivial way.

If B is not valid, then $\emptyset$ does not strongly sc-entail B. At the same time $X \neq \emptyset$, B is consistent and '$\neg B$' $\notin X$.

For conciseness, we write $\mathsf{card}(X) = n$ instead of "X has n elements."

If $\mathsf{card}(X) = 1$, then X itself is the only set looked for.

Assume that $\mathsf{card}(X) = n$, where $n > 1$. We define the following families of sets of wffs:

$$\Upsilon_1 = \{Z : Z = \{A, \neg B\},\ where\ A \in X\} \tag{9.6}$$

$$\Upsilon_n = \{X, \neg B\} \tag{9.7}$$

For each j, where $1 < j < n$, we define the corresponding family of sets Υ_j of wffs as follows:

$$\Upsilon_j = \{Z : Z = \{A_1, \ldots, A_j, \neg B\},\ \textit{where}\ \{A_1, \ldots, A_j\} \subseteq X\} \tag{9.8}$$

Note that each element of Υ_k, where $1 \leqslant k \leqslant n$, can be displayed as:

$$Z' \cup \{\neg B\} \tag{9.9}$$

where $Z' \subseteq X$ and $\mathsf{card}(Z') = k$. As '$\neg B$' $\notin X$, for any $Y \in \Upsilon_k$ we have $\mathsf{card}(Y) = k + 1$.

We consider the following sequence of families of sets of wffs:

$$\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n \tag{9.10}$$

At this point we aim at identifying all the elements of the following family of sets:

$$\bigcup_{i=1}^{i=n} \Upsilon_i \tag{9.11}$$

that are MI-sets. (By Theorem 8.2, at least one element of (9.11) is a MI-set.) The procedure sketched below solves the problem by performing a series of consistency checks.

Step 1. We consider the elements of Υ_1 one after another in order to check whether a given element is consistent; it is clear that each inconsistent element of Υ_1 is a MI-set of the required kind.[58] The outcome is:

$$\langle \Sigma_1, \Gamma_1 \rangle$$

where Σ_1 is the set of all the inconsistent elements of Υ_1 (and hence MI-sets). If there are no such elements, $\Sigma_1 = \emptyset$; otherwise $\Sigma_1 \neq \emptyset$. Let:

$$\Sigma_1^{\nearrow} =_{df} \{Y \in \Upsilon_2 : W \subseteq Y\ \textit{for some}\ W \in \Sigma_1\}$$

Γ_1 is defined as follows.

1. *If* $\Sigma_1 = \Upsilon_1$, *then* $\Gamma_1 = \emptyset$.
2. *If* $\Sigma_1 \neq \Upsilon_1$, *then* $\Gamma_1 = (\Upsilon_2 \setminus \Sigma_1^{\nearrow})$.

[58] Since B is not valid, '$\neg B$' is not inconsistent. X is, by assumption, a consistent set, so each wff in X is consistent.

We stop if $\Gamma_1 = \emptyset$, otherwise we go to Step 2, at which we consider all the elements of Γ_1 by checking their consistency.

A comment is in order. When $\Sigma_1 = \Upsilon_1$, each set of the form $\{A, \neg B\}$, where $A \in X$, happens to be a MI-set and hence the family (9.11) contains no further MI-sets. If $\Sigma_1 \neq \Upsilon_1$ and $\Sigma_1 \neq \emptyset$, at the second step we consider only the elements of Υ_2 which belong to Γ_1. This is due to the fact that each element of $\Sigma_1^{\nearrow}$ is an inconsistent set which has an inconsistent proper subset. So there is no reason to consider the elements of Υ_2 that belong to $\Sigma_1^{\nearrow}$, since we know in advance that none of them is a MI-set. So we consider the set $\Upsilon_2 \setminus \Sigma_1^{\nearrow}$ only. If, however, $\Sigma_1 = \emptyset$, we have $\Sigma_1^{\nearrow} = \emptyset$ and thus $\Gamma_1 = \Upsilon_2$, that is, the whole set Υ_2 is taken into consideration at the next step.

Assume that we did not stop at Step i, i.e., that $\Gamma_i \neq \emptyset$.

Step i+1. We check the consistency of the consecutive elements of the set Γ_i obtained at step i; recall that $\Gamma_i \subseteq \Upsilon_{i+1}$. The outcome is:

$$\langle \Sigma_{i+1}, \Gamma_{i+1} \rangle$$

where Σ_{i+1} is the set of all the inconsistent elements of Γ_i, if there are any, and $\Sigma_{i+1} = \emptyset$ otherwise, while Γ_{i+1} is defined by the conditions:

1. *If* $\Sigma_{i+1} = \Gamma_i$, *then* $\Gamma_{i+1} = \emptyset$.
2. *If* $\Sigma_{i+1} \neq \Gamma_i$, *then* $\Gamma_{i+1} = (\Upsilon_{i+2} \setminus \Sigma_{i+1}^{\nearrow})$.

where

$$\Sigma_{i+1}^{\nearrow} =_{df} \{Y \in \Upsilon_{i+2} : W \subseteq Y \textit{ for some } W \in \Sigma_i\}$$

We stop if $\Gamma_{i+1} = \emptyset$; otherwise we go to Step $i+2$.

Observe that the elements of Σ_{i+1}, if there are any, are MI-sets. To see this, assume that $Y \in \Sigma_{i+1}$. Thus Y has been selected from the elements of Γ_i, that is, $Y \in \Gamma_i$. Suppose that Y has inconsistent proper subset(s). As Y is finite, the number of its inconsistent proper subsets is also finite. So there exists an index $e < i+1$ such that some inconsistent proper subset of Y, say, Y^*, has exactly e elements and no inconsistent proper subset of Y has less than e elements. But each proper subset of Y^* has at most $e-1$ elements and is a proper subset of Y. Hence Y^* is a MI-set included in Y such that $\mathsf{card}(Y^*) = e$. As such, Y^* belongs to Σ_{e-1}. It follows that:

$$Y \in \Sigma_{e-1}^{\nearrow}$$

and hence $Y \notin \Gamma_i$. A contradiction. Therefore Y has no inconsistent proper subset. But Y is inconsistent, because $Y \in \Sigma_{i+1}$. Hence Y is a MI-set.

Since X is a finite non-empty set, it is clear that the procedure sketched above terminates in a finite number of steps and produces the required result. Observe that when $\Sigma_{n-1} = \emptyset$, the set looked for is X itself. However, we arrive at this point when consistency checks of all sets of the form $Z \cup \{\neg B\}$, where Z is a non-empty proper subset of X, have produced affirmative results. There are $2^n - 2$ such sets. Hence the procedure described above provides the required result after performing at most $2^n - 2$ consistency checks. This is the "worst case" possible.

Chapter 10

Towards Proof Theory for Strong Entailments

10.1 Introduction

Both strong multiple-conclusion entailment and strong single-conclusion entailment are subrelations of the respective "standard" entailments relations and thus of the corresponding consequence relations. Strong entailments are defined in semantic terms. Is it possible characterize strong entailments proof-theoretically? In this chapter I provide an affirmative answer to this question, yet restricted to the classical propositional case.

As Theorem 6.5 shows, a problem of the form:

(P) *Does X strongly mc-entail Y?*

splits into two sub-problems:

(P_1) *Is it the case that $X \cap \neg Y = \emptyset$?*

(P_2) *Is $X, \neg Y$ a* MI*-set?*

Similarly, due to Theorem 7.1, a problem of the form:

(P') *Does X strongly sc-entail A?*

splits into:

(P_1') *Is it the case that '$\neg A$' $\notin X$?*

(P_2') *Is $X, \neg A$ a* MI*-set?*

Questions (P_1) and (P_1’) can be resolved by syntactic means. But either (P_2) or (P_2’) is a problem that pertains to a semantic property. In order to solve it syntactically one needs a proof-theoretic account of MI-sets.

10.2 Calculating MI-sets by Means of the System $\mathsf{HI}^{\mathsf{CPL}}$

The system $\mathsf{HI}^{\mathsf{CPL}}$ presented in Chapter 3 provides a proof-theoretic account of finite HI-sets (that is, holistically inconsistent sets; cf. Definition 3.4) of CPL-wffs. However, the system can be employed, in a somehow tricky way, to calculate MI-sets of CPL-wffs. This is due to the fact that a finite MI-set can be represented by the corresponding HI-set.

Recall that, due to Theorem 6.3, any MI-set of CPL-wffs is non-empty and finite. Let $Z = \{C_1, \ldots, C_n\}$. We assign to Z a set $Z^\bullet$, defined in the following way:

Definition 10.1. *Let* $Z = \{C_1, \ldots, C_n\}$.

1. *If* $n = 1$, *then* $Z^\bullet = \{C_1\}$.
2. *If* $n > 1$, *then*

$$Z^\bullet = \{C_2 \wedge \ldots \wedge C_n, C_1 \wedge \cdots \wedge C_{j-1} \wedge C_{j+1} \wedge \ldots \wedge C_n, \ldots, C_1 \wedge \ldots \wedge C_{n-1}\}.$$

The following holds:

Corollary 10.1. *If* $Z = \{C_1, \ldots, C_n\}$, *then* Z *is a* MI*-set iff* $Z^\bullet$ *is a* HI*-set.*

Proof. ($\Rightarrow$) If Z is a MI-set, then Z is inconsistent. Suppose that $Z^\bullet$ is consistent. It follows that Z is consistent. Thus $Z^\bullet$ is inconsistent.

As Z is a MI-set, each proper subset of Z is consistent. So each element of $Z^\bullet$ is consistent.

Therefore $Z^\bullet$ is a HI-set.

($\Leftarrow$) If $Z^\bullet$ is a HI-set, then $Z^\bullet$ is inconsistent. As $Z^\bullet$ is a HI-set, all the wffs in $Z^\bullet$ are consistent and this amounts to the consistency of all proper subsets of Z. □

Thus in order to establish that Z is a MI-set it suffices to provide a $\mathsf{HI}^{\mathsf{CPL}}$-proof of a sequent which has all the elements of $Z^\bullet$ left to the turnstile, $\vdash$.

However, the system $\mathsf{HI}^{\mathsf{CPL}}$ “calculates” also HI-sets which do not represent any MI-set. Moreover, proofs of sequents in $\mathsf{HI}^{\mathsf{CPL}}$ are rather tedious. So let me turn to another system.

10.3 The Calculus $\mathsf{MI}^{\mathsf{CPL}}$

In this section I present a calculus, dubbed $\mathsf{MI}^{\mathsf{CPL}}$, in which provable sequents of a strictly defined form correspond to MI-sets. In the case of $\mathsf{MI}^{\mathsf{CPL}}$, however, sequents are construed differently than in $\mathsf{HI}^{\mathsf{CPL}}$ (see below).

Rules of $\mathsf{MI}^{\mathsf{CPL}}$ operate on sequences of sequents. Since a sequence of sequents is customarily called a hypersequent, $\mathsf{MI}^{\mathsf{CPL}}$ may be called a calculus of hypersequents. But speaking about hypersequent calculi usually brings into mind Avron's seminal works.[59] However, the format of $\mathsf{MI}^{\mathsf{CPL}}$ differs considerably from that of Avron-style hypersequent calculi. In particular, derivations and proofs in $\mathsf{MI}^{\mathsf{CPL}}$ are not trees having hypersequents in their nodes, but sequences of hypersequents. Rules of $\mathsf{MI}^{\mathsf{CPL}}$ transform hypersequents into hypersequents, and a rule is always applied to the last term of a derivation constructed so far. Last but not least, $\mathsf{MI}^{\mathsf{CPL}}$ has no axioms, but comprises rules only.

Given these substantial differences, and taking into account that the concept of hypersequent is loaded with references to Avron-style calculi, let me use a new term for a sequence of sequents. The term chosen is "seqsequent," after Latin *sequentia*, which means (among others) "sequence."[60] No doubt, saying that we aim at a calculus of seqsequents is less misleading than speaking about a calculus of hypersequents. A warning is needed, however. As we will see, the order in which sequents occur in a "seqsequent" does not determine the order of application of rules of the calculus. We are speaking about a calculus of seqsequents only to stress that rules of the calculus operate on "seqsequents," that is, a rule transforms a sequence of sequents into a sequence of sequents.

10.3.1 Numerically Annotated Wffs, Sequents, Seqsequents, and More

We will be operating with sequents based on sequences of numerically annotated wffs.

[59] Starting from the influential paper [2]. Avron-style approach is not the only one, however. A reader interested in different types of hypersequent calculi (including those in which hypersequents are construed as sets or multisets of sequents rather than their sequences) is advised to consult [23], Chapter 4.7.

[60] "Seq" is not a prefix in English, but since many English words begin with prefixes rooted in Latin, I hope that this proposal is acceptable. A reader familiar with programming is kindly requested to suspend any associations he/she may have.

Definition 10.2 (Numerically annotated wff; na-wff). *A numerically annotated wff (na-wff for short) is an expression of the form* $A^{[i]}$, *where* A *is a wff and* i *is a numeral from the set* $\{1, 2, 3, \ldots\}$.

Let us stress that numerals are here proof-theoretic devices only. It is not assumed that they refer to possible worlds or perform the function of labels.

Notation In order to keep the number of brackets to a minimum, the following notational convention is adopted: an inscription of the form $\neg A^{[i]}$ refers to a wff of the form $\neg A$ annotated with numeral i.

From now on, the term "sequent" will be understood in the following way:

Definition 10.3 (Sequent). *A sequent is an expression of the form:*

$$C_1^{[i_1]}, \ldots, C_m^{[i_m]} \vdash \tag{10.1}$$

where $C_1^{[i_1]}, \ldots, C_m^{[i_m]}$ *is a finite sequence of na-wffs.*

When $m = 0$, we write the corresponding sequent as $\emptyset \vdash$. Although we consider sequents with empty succedents, it is no accident that we put the turnstile $\vdash$ into a sequent. This will allow us to differentiate between operations on sequents and operation on sequences of annotated wffs (see below).

Remark 10.1. Note that sequents used in $\mathsf{MI}^{\mathsf{CPL}}$ differ from these used in $\mathsf{HI}^{\mathsf{CPL}}$. The former have sequences of numerically annotated wffs to the left of the turnstile, while the latter have sets of wffs on the left to the turnstile.

We need the concepts of atomic sequent, closed atomic sequent, and open atomic sequent.

Definition 10.4 (Atomic sequent). *An atomic sequent is an expression of the form:*

$$l_1^{[i_1]}, \ldots, l_m^{[i_m]} \vdash \tag{10.2}$$

where $l_1, \ldots, l_m$ *are literals, that is, propositional variables or their negations.*

Definition 10.5 (Open atomic sequent and closed atomic sequent). *An atomic sequent:*

$$l_1^{[i_1]}, \ldots, l_m^{[i_m]} \vdash$$

is closed if it involves na-wffs based on complementary literals, i.e. there exist $l_j^{[i_j]}, l_k^{[i_k]}$ $(1 \leqslant j, k \leqslant m)$ *such that* $l_j = \neg l_k$. *An atomic sequent is open if it is not closed.*

We use the letters $\rho, \sigma, \theta, \xi, \zeta$, possibly with subscripts, as metalanguage variables for finite sequences of na-wffs, the empty sequence included.

Some further technical concepts are needed. The expression `Form` refers to the set of CPL-wffs.

Definition 10.6 (The set of wffs of a sequent)**.** *Let* $\sigma \vdash$ *be a sequent.*

$$\mathsf{wff}(\sigma \vdash) = \{A \in \texttt{Form} : A^{[i]} \textit{ is a term of } \sigma\}.$$

Definition 10.7 (Withdrawal)**.** *Let* σ *be a finite sequence of na-wffs. By* $f_{\setminus[i_j]}(\sigma)$ *we mean the subsequence of* σ *resulting from it by removing all its terms (i.e. na-wffs) which are annotated with the numeral* i_j.

Needless to say, if $\sigma \vdash$ is a sequent, so is $f_{\setminus[i_j]}(\sigma) \vdash$.[61]

Definition 10.8 (Seqsequent)**.** *A seqsequent is a finite sequence of sequents.*

In this chapter we use the letters Φ, Ψ, Γ, with subscripts or superscriptswhen necessary, as metalanguage variables for seqsequents.

Definition 10.9 (Constituent of a seqsequent)**.** *By a constituent of a seqsequent we mean any sequent which is a term of the seqsequent.*

Finally, we distinguish *ordered sequents.*

Definition 10.10 (Ordered sequent)**.** *An ordered sequent is a sequent which falls under the schema:*

$$C_1^{[1]}, \ldots, C_m^{[m]} \vdash \tag{10.3}$$

where $m \geqslant 1$, *and* $C_1, \ldots, C_m$ *are pairwise syntactically distinct wffs when* $m > 1$.

[61] If i_j is the only numeral which occurs in na-wffs of σ, then $f_{\setminus[i_j]}(\sigma) \vdash$ equals $\emptyset \vdash$, which is, by definition, a sequent. If $\sigma \vdash = \emptyset \vdash$, then $f_{\setminus[i_j]}(\sigma) \vdash = \emptyset \vdash$. Of course, $\mathsf{wff}(\emptyset \vdash) = \emptyset$.

Thus, besides sequents of the form $A^{[1]} \vdash$, ordered sequents are sequents whose consecutive terms (with the exception of the turnstile), are pairwise syntactically distinct wffs annotated with consecutive numerals (occurring in curly brackets), starting from the numeral 1.[62] At the metalanguage level, ordered sequents of the form (10.3) will be concisely written as:

$$C_{\vec{m}}^{[\vec{m}]} \vdash \tag{10.4}$$

As we will see, in order to show that $\{C_1, \ldots, C_m\}$ is a MI-set it suffices to prove the corresponding ordered sequent $C_{\vec{m}}^{[\vec{m}]} \vdash$.

10.3.2 Rules and Proofs

In order to present the rules of $\mathsf{MI}^{\mathsf{CPL}}$ in a concise manner let us introduce some notational conventions first.

Following [46], we distinguish between α-wffs and β-wffs, and we assign two further wffs to any of them. This is explained in Table 10.1 below.

α	α_1	α_2	β	β_1	β_2
$A \wedge B$	A	B	$\neg(A \wedge B)$	$\neg A$	$\neg B$
$\neg(A \vee B)$	$\neg A$	$\neg B$	$A \vee B$	A	B
$\neg(A \rightarrow B)$	A	$\neg B$	$A \rightarrow B$	$\neg A$	B

Table 10.1: *α-wffs and β-wffs.*

We use the sign $'$ as the concatenation-sign for sequences of na-wffs. For brevity, we assume that a metalanguage expression of the form $\sigma\ 'A^{[i]}$ denotes the concatenation of sequence σ and the one-term sequence $\langle A^{[i]} \rangle$, while a metalanguage expression of the form $\sigma\ 'A^{[i]}\ '\ \theta$ refers to the concatenation of sequence $\sigma\ 'A^{[i]}$ and sequence θ.

The semicolon will perform the role of the concatenation-sign for seqsequents. We usually omit angle brackets when referring to a seqsequent which has only one constituent. Thus $\Psi;\ \sigma \vdash$ stands for the concatenation of Ψ and $\langle \sigma \vdash \rangle$. The expression $\Psi;\ \sigma \vdash;\ \Phi$ refers to the concatenation of $\Psi;\ \sigma \vdash$ and Φ.

The calculus $\mathsf{MI}^{\mathsf{CPL}}$ has only rules which operate on seqsequents. No axioms are provided. Here are the primary rules of $\mathsf{MI}^{\mathsf{CPL}}$:

[62] Each ordered sequent is a sequent, but not the other way round. For instance, the expressions $p^{[4]}, q^{[2]} \vdash$ and $p^{[1]}, p^{[3]} \vdash$ are sequents in our sense, but none of them is an ordered sequent. Similarly, $p^{[1]}, p^{[1]} \vdash$ and $p^{[1]}, p^{[2]} \vdash$ are sequents, though neither of them is an ordered sequent.

$$\frac{\Phi;\ \sigma \,'\, \alpha^{[i]} \,'\, \theta\ \vdash;\ \Psi}{\Phi;\ \sigma \,'\, \alpha_1^{[i]} \,'\, \alpha_2^{[i]} \,'\, \theta \vdash;\ \Psi} \qquad (\mathsf{R}_\alpha^{[i]})$$

$$\frac{\Phi;\ \sigma \,'\, \beta^{[i]} \,'\, \theta \vdash;\ \Psi}{\Phi;\ \sigma \,'\, \beta_1^{[i]} \,'\, \theta \vdash;\ \sigma \,'\, \beta_2^{[i]} \,'\, \theta \vdash;\ \Psi} \qquad (\mathsf{R}_\beta^{[i]})$$

$$\frac{\Phi;\ \sigma \,'\, \neg\neg A^{[i]} \,'\, \theta \vdash;\ \Psi}{\Phi;\ \sigma \,'\, A^{[i]} \,'\, \theta \vdash;\ \Psi} \qquad (\mathsf{R}_{\neg\neg}^{[i]})$$

Any of Φ, Ψ, σ, θ can be empty.

Observe that rules of $\mathsf{MI}^{\mathsf{CPL}}$ "act locally": if a rule is applied to a seqsequent, only one constituent and only one occurrence of a na-wff in the constituent are acted upon, while the other occurrences and other constituents remain unaffected. Moreover, any new na-wff that comes into play due to an application of a rule is annotated with the same numeral as the na-wff acted upon.

We are now ready for an introduction of the concept of proof of an ordered sequent.

Definition 10.11 ($\mathsf{MI}^{\mathsf{CPL}}$-proof of an ordered sequent)**.** *A finite sequence of seqsequents $\Gamma_1, \ldots, \Gamma_n$ is a $\mathsf{MI}^{\mathsf{CPL}}$-proof of an ordered sequent $C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash$ iff*

1. $\Gamma_1 = \langle C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash \rangle$,
2. *Γ_{j+1} results from Γ_j by a rule of $\mathsf{MI}^{\mathsf{CPL}}$, where $1 \leqslant j < n$,*
3. *each constituent of Γ_n is a closed atomic sequent,*
4. *for each $k \in \{1, \ldots, m\}$ there exists a constituent $\sigma \vdash$ of Γ_n such that the sequent $f_{\backslash[k]}(\sigma) \vdash$ is an open atomic sequent or is of the form $\emptyset \vdash$.*

An ordered sequent is provable in $\mathsf{MI}^{\mathsf{CPL}}$ iff the sequent has a $\mathsf{MI}^{\mathsf{CPL}}$-proof.

Remark 10.2. The concept of proof introduced above is non-standard in many respects. First, a proof is a sequence of seqsequents. Second, notice that it is an ordered sequent (i.e. a sequent in which wffs occurring left of the turnstile are annotated with consecutive numerals, starting from 1) that performs the role of an "input" of a proof: the first line of a proof is a one-term seqsequent involving an ordered sequent. We do not introduce the concept of proof of a sequent in general, but only of an ordered sequent. As we will see, this is sufficient for our purposes. Third, $\mathsf{MI}^{\mathsf{CPL}}$-proofs in are strictly linear: Γ_{j+1} results by a rule from Γ_j only.

As we will show (cf. Theorem 10.1 below), the existence of a $\mathsf{MI}^{\mathsf{CPL}}$-proof of an ordered sequent $\sigma \vdash$ ensures that $\mathsf{wff}(\sigma \vdash)$ is a MI-set of CPL-wffs.

10.3.3 Examples of Proofs

Example 10.26. The one-term sequence

$$\langle\langle p^{[1]}, \neg p^{[2]} \vdash\rangle\rangle$$

is a proof of the ordered sequent:

$$p^{[1]}, \neg p^{[2]} \vdash \tag{10.5}$$

since $f_{\setminus[1]}(p^{[1]}, \neg p^{[2]}) \vdash\, = \neg p^{[2]} \vdash$ and $f_{\setminus[2]}(p^{[1]}, \neg p^{[2]}) \vdash\, = p^{[1]} \vdash$.

For brevity, in what follows we will be omitting angle brackets in the first line of a proof, and in the case of one-term seqsequents.

Example 10.27. Here is a proof of the ordered sequent:

$$(p \wedge \neg p)^{[1]} \vdash \tag{10.6}$$

(Inscriptions of the form $\mathsf{R}_x^{[i]}$ do not belong to proofs, but indicate what rule has been applied to the uppermost seqsequent. Horizontal lines separate terms of a proof.)

$$\frac{(p \wedge \neg p)^{[1]} \vdash}{p^{[1]}, \neg p^{[1]} \vdash}\ \mathsf{R}_\alpha^{[1]}$$

Notice that $f_{\setminus[1]}(p^{[1]}, \neg p^{[1]} \vdash)$ equals $\emptyset \vdash$.

Example 10.28. The following:

$$\frac{p^{[1]}, \neg(\neg p \to q)^{[2]} \vdash}{p^{[1]}, \neg p^{[2]}, \neg q^{[2]} \vdash}\ \mathsf{R}_\alpha^{[2]}$$

is a proof of the ordered sequent:

$$p^{[1]}, \neg(\neg p \to q)^{[2]} \vdash \tag{10.7}$$

For the sake of transparency, from now on we highlight the na-wff on which the rule indicated to the right acts upon. We tick exemplary occurrences of numerals due to which clause (*4*) of the definition of proof is satisfied.

Example 10.29. Here is a proof of the ordered sequent:

$$(p \vee q)^{[1]}, \neg p^{[2]}, \neg q^{[3]}, \vdash \tag{10.8}$$

$$\frac{\boxed{(p \vee q)^{[1]}}\,, \neg p^{[2]}, \neg q^{[3]} \vdash}{p^{[1\checkmark]}, \neg p^{[2\checkmark]}, \neg q^{[3]} \vdash;\ \ q^{[1]}, \neg p[2], \neg q^{[3\checkmark]} \vdash}\ \mathsf{R}_\beta^{[3]}$$

Example 10.30. A proof of the ordered sequent:

$$(p \to q)^{[1]}, p^{[2]}, \neg q^{[3]} \vdash \tag{10.9}$$

$$\frac{\boxed{(p \to q)^{[1]}}\,, p^{[2]}, \neg q^{[3]} \vdash}{\neg p^{[1\checkmark]}, p^{[2\checkmark]}, \neg q^{[3]} \vdash;\ \ q^{[1]}, p^{[2]}, \neg q^{[3\checkmark]} \vdash}\ \mathsf{R}_\beta^{[1]}$$

Example 10.31. The following is a proof of the ordered sequent:

$$(p \wedge (\neg p \vee q))^{[1]}, \neg q^{[2]} \vdash \tag{10.10}$$

$$\frac{\dfrac{\boxed{(p \wedge (\neg p \vee q))^{[1]}}\,, \neg q^{[2]} \vdash}{p^{[1]},\ \boxed{(\neg p \vee q)^{[1]}}\,, \neg q^{[2]} \vdash}\ \mathsf{R}_\alpha^{[1]}}{p^{[1\checkmark]}, \neg p^{[1]}, \neg q^{[2]} \vdash;\ \ p^{[1]}, q^{[1]}, \neg q^{[2\checkmark]} \vdash}\ \mathsf{R}_\beta^{[1]}$$

Example 10.32. A proof of the ordered sequent:

$$(p \to (q \to r))^{[1]}], (p \wedge q)^{[2]}, \neg r^{[3]} \vdash \tag{10.11}$$

$$\frac{\dfrac{\dfrac{(p \to (q \to r))^{[1]}], \boxed{(p \wedge q)^{[2]}}\,, \neg r^{[3]} \vdash}{\boxed{(p \to (q \to r))^{[1]}}\,, p^{[2]}, q^{[2]}, \neg r^{[3]} \vdash}\ \mathsf{R}_\alpha^{[1]}}{\neg p^{[1]}, p^{[2]}, q^{[2]}, \neg r^{[3]} \vdash;\ \ \boxed{(q \to r)^{[1]}}\,, p^{[2]}, q^{[2]}, \neg r^{[3]} \vdash}\ \mathsf{R}_\beta^{[1]}}{\neg p^{[1\checkmark]}, p^{[2]}, q^{[2]}, \neg r^{[3]} \vdash;\ \ \neg q^{[1]}, p^{[2]}, q^{[2\checkmark]}, \neg r^{[3]} \vdash;\ \ r^{[1]}, p^{[2]}, q^{[2]}, \neg r^{[3\checkmark]} \vdash}\ \mathsf{R}_\beta^{[1]}$$

Example 10.33. A proof of the ordered sequent:

$$p \vee (q \vee r))^{[1]}, \neg(p \vee q)^{[2]}, \neg(p \vee r)^{[3]} \vdash \tag{10.12}$$

$$\begin{array}{cl}
p \lor (q \lor r))^{[1]}, \neg(p \lor q)^{[2]}, \neg(p \lor r)^{[3]} \vdash & \mathsf{R}_\alpha^{[3]} \\ \hline
(p \lor (q \lor r))^{[1]}, \neg(p \lor q)^{[2]}, \neg p^{[3]}, \neg r^{[3]} \vdash & \mathsf{R}_\alpha^{[2]} \\ \hline
(p \lor (q \lor r))^{[1]}, \neg p^{[2]}, \neg q^{[2]}, \neg p^{[3]}, \neg r^{[3]} \vdash & \mathsf{R}_\beta^{[1]} \\ \hline
p^{[1]}, \neg p^{[2]}, \neg q^{[2]}, \neg p^{[3]}, \neg r^{[3]} \vdash ;\ (q \lor r)^{[1]}, \neg p^{[2]}, \neg q^{[2]}, \neg p^{[3]}, \neg r^{[3]} \vdash & \mathsf{R}_\beta^{[1]} \\ \hline
p^{[1\checkmark]}, \neg p^{[2]}, \neg q^{[2]}, \neg p^{[3]}, \neg r^{[3]} \vdash ;\ q^{[1]}, \neg p^{[2]}, \neg q^{[2\checkmark]}, \neg p^{[3]}, \neg r^{[3]} \vdash ;\ r^{[1]}, \neg p^{[2]}, \neg q^{[2]}, \neg p^{[3]}, \neg r^{[3\checkmark]} \vdash &
\end{array}$$

Example 10.34. A proof of the ordered sequent:

$$(p \to (q \to r))^{[1]}, (p \to q)^{[2]}, \neg(p \to r)^{[3]} \vdash \tag{10.13}$$

$$\begin{array}{cl}
(p \to (q \to r))^{[1]}, (p \to q)^{[2]}, \neg(p \to r)^{[3]} \vdash & \mathsf{R}_\alpha^{[3]} \\ \hline
(p \to (g \to r))^{[1]}, (p \to q)^{[2]}, p^{[3]}, \neg r^{[3]} \vdash & \mathsf{R}_\beta^{[2]} \\ \hline
(p \to (q \to r))^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash ;\ (p \to (q \to r))^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash & \mathsf{R}_\beta^{[1]} \\ \hline
\neg p^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash ;\ (q \to r)^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash ;\ (p \to (q \to r))^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash & \mathsf{R}_\beta^{[1]} \\ \hline
\neg p^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash ;\ \neg q^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash ;\ r^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash ;\ (p \to (q \to r))^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash & \mathsf{R}_\beta^{[1]} \\ \hline
\neg p^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash ; \neg q^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash ;\ r^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash ; \neg p^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash ;\ (q \to r)^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash & \mathsf{R}_\beta^{[1]} \\ \hline
\neg p^{[1]}, \neg p^{[2]}, p^{[3\checkmark]}, \neg r^{[3]} \vdash ;\ \neg q^{[1]}, \neg p^{[2\checkmark]}, p^{[3]}, \neg r^{[3]} \vdash ;\ r^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash ;\ \neg p^{[1\checkmark]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash ;\ \neg q^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash ;\ r^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash &
\end{array}$$

10.4 Soundness and Completeness

10.4.1 Some Lemmas

In order to prove soundness and completeness of the calculus $\mathsf{MI}^{\mathsf{CPL}}$ with respect to MI-sets we need a series of corollaries and lemmas.

As an immediate consequence of definitions introduced in sections 10.3.1 and 10.3.2 we get:

Corollary 10.2. *X is a MI-set iff there exists an ordered sequent $C_{\vec{m}}^{[\vec{m}]} \vdash$ such that $X = \mathsf{wff}(C_{\vec{m}}^{[\vec{m}]} \vdash)$ and*

1. *$\mathsf{wff}(C_{\vec{m}}^{[\vec{m}]} \vdash)$ is an inconsistent set, and*
2. *for each $k \in \{1, \ldots, m\}$: the set $\mathsf{wff}(f_{\backslash[k]}(C_{\vec{m}}^{[\vec{m}]} \vdash))$ is consistent.*

The following hold:

Lemma 10.1.

1. *$\mathsf{wff}(\sigma \ ' \ \alpha^{[i]} \ ' \ \theta \vdash)$ is inconsistent iff $\mathsf{wff}(\sigma \ ' \ \alpha_1^{[i]} \ ' \ \alpha_2^{[i]} \ ' \ \theta \vdash)$ is inconsistent.*
2. *$\mathsf{wff}(\sigma \ ' \ \beta^{[i]} \ ' \ \theta \vdash)$ is inconsistent iff $\mathsf{wff}(\sigma \ ' \ \beta_1^{[i]} \ ' \ \theta \vdash)$ is inconsistent and $\mathsf{wff}(\sigma \ ' \ \beta_2^{[i]} \ ' \ \theta \vdash)$ is inconsistent.*
3. *$\mathsf{wff}(\sigma \ ' \ A^{[i]} \ ' \ \theta \vdash)$ is inconsistent iff $\mathsf{wff}(\sigma \ ' \ \neg\neg A^{[i]} \ ' \ \theta \vdash)$ is inconsistent.*

Lemma 10.2.

1. *If $\Phi; \ \sigma \vdash; \ \Psi$ results from $\Phi; \ \theta \vdash; \ \Psi$ by a rule of $\mathsf{MI}^{\mathsf{CPL}}$, then the set $\mathsf{wff}(\sigma \vdash)$ is inconsistent iff $\mathsf{wff}(\theta \vdash)$ is an inconsistent set.*
2. *If $\Phi; \ \sigma_1 \vdash; \ \sigma_2 \vdash; \ \Psi$ results from $\Phi; \ \theta \vdash; \ \Psi$ by a rule of $\mathsf{MI}^{\mathsf{CPL}}$, then both $\mathsf{wff}(\sigma_1 \vdash)$ and $\mathsf{wff}(\sigma_2 \vdash)$ are inconsistent sets iff $\mathsf{wff}(\theta \vdash)$ is an inconsistent set.*

Proof. If $\Phi; \ \sigma \vdash; \ \Psi$ results from $\Phi; \ \theta \vdash; \ \Psi$ by a rule of $\mathsf{MI}^{\mathsf{CPL}}$, then θ involves a numerically annotated α-wff or a numerically annotated double negated wff. But $X, \alpha \Vdash \emptyset$ iff $X, \alpha_1, \alpha_2 \Vdash \emptyset$, and $X, \neg\neg A \Vdash \emptyset$ iff $X, A \Vdash \emptyset$.

If $\Phi; \ \sigma_1 \vdash \ \sigma_2 \vdash; \ \Psi$ results from $\Phi; \ \theta \vdash; \ \Psi$ by a rule of $\mathsf{MI}^{\mathsf{CPL}}$, then a numerically annotated β-wff is a term of θ. Yet, $X, \beta \Vdash \emptyset$ iff $X, \beta_1 \Vdash \emptyset$ and $X, \beta_2 \Vdash \emptyset$. □

Lemma 10.3. *If a seqsequent Ψ results from a seqsequent Φ by a rule of* $\mathsf{MI}^{\mathsf{CPL}}$*, then the following conditions are equivalent:*

1. *for each constituent $\sigma \vdash$ of Φ: the set* $\mathsf{wff}(\sigma \vdash)$ *is inconsistent,*
2. *for each constituent $\theta \vdash$ of Ψ: the set* $\mathsf{wff}(\theta \vdash)$ *is inconsistent.*

Proof. By Lemma 10.2. □

10.4.2 Soundness w.r.t. MI-sets

Provability in $\mathsf{MI}^{\mathsf{CPL}}$ and the property of being a MI-set are linked in a way characterized by:

Theorem 10.1 (Soundness w.r.t. MI-sets)**.** *Let X be a finite non-empty set of wffs, and let $\sigma \vdash$ be an ordered sequent such that* $\mathsf{wff}(\sigma \vdash) = X$*. If the sequent $\sigma \vdash$ is provable in* $\mathsf{MI}^{\mathsf{CPL}}$*, then X is a* MI*-set.*

Proof. Let $C^{[\overrightarrow{m}]}_{\overrightarrow{m}} \vdash$ be an arbitrary but fixed ordered sequent such that $X = \mathsf{wff}(C^{[\overrightarrow{m}]}_{\overrightarrow{m}} \vdash)$. Assume that

$$\Gamma_1, \ldots, \Gamma_n \tag{10.14}$$

is a $\mathsf{MI}^{\mathsf{CPL}}$-proof of the sequent $C^{[\overrightarrow{m}]}_{\overrightarrow{m}} \vdash$. By Definition 10.11, each constituent of Γ_n is a closed atomic sequent. Hence the set $\mathsf{wff}(\theta \vdash)$ is inconsistent for each constituent $\theta \vdash$ of Γ_n. Therefore, by Lemma 10.3, the set $\mathsf{wff}(C^{[\overrightarrow{m}]}_{\overrightarrow{m}} \vdash)$ is inconsistent, that is, X is inconsistent.

We shall prove the following:

(★) *if Γ_{j+1} has a constituent, $\sigma \vdash$, such that the set*

$$\mathsf{wff}(f_{\backslash [k]}(\sigma) \vdash)$$

is consistent, then Γ_j has a constituent, θ, such that the set

$$\mathsf{wff}(f_{\backslash [k]}(\theta) \vdash)$$

is consistent, where $1 \leqslant j < n$ and $1 \leqslant k \leqslant m$.

Let $\sigma \vdash$ be a constituent of Γ_{j+1} for which the set $\mathsf{wff}(f_{\backslash [k]}(\sigma) \vdash)$ is consistent. Recall that rules of $\mathsf{MI}^{\mathsf{CPL}}$ "act locally": if a rule is applied to a seqsequent, only one constituent and only one occurrence of a na-wff in the constituent are acted upon (more precisely, only one term of the sequence of na-wffs which occurs in the constituent is transformed). When Γ_{j+1} results from Γ_j by a rule, the following cases are possible:

(a) $\sigma \vdash$ has been rewritten from Γ_j into Γ_{j+1} (since a rule has been applied to Γ_j w.r.t. some other constituent of it),

(b) the occurrence of $\sigma \vdash$ in Γ_{j+1} is due to an application of a rule to Γ_j w.r.t. a constituent, say, $\theta \vdash$, of Γ_j.

If (a) is the case, then (★) holds trivially. So assume that (b) holds. Two sub-cases are possible:

(b_1) a rule has been applied to Γ_j w.r.t. the constituent $\theta \vdash$ and a term of θ annotated with k,

(b_2) a rule has been applied to Γ_j w.r.t. the constituent $\theta \vdash$ and a term of θ which is annotated with some numeral j different from k.

If (b_1) holds, then $\mathsf{wff}(f_{\setminus[k]}(\theta) \vdash) = \mathsf{wff}(f_{\setminus[k]}(\sigma) \vdash)$, so the set

$$\mathsf{wff}(f_{\setminus[k]}(\theta) \vdash)$$

is consistent. Assume that (b_2) is the case. Suppose that the set $\mathsf{wff}(f_{\setminus[k]}(\theta) \vdash)$ is inconsistent though $\mathsf{wff}(f_{\setminus[k]}(\sigma) \vdash)$ is a consistent set. Both $\mathsf{wff}(f_{\setminus[k]}(\sigma) \vdash)$ and $\mathsf{wff}(f_{\setminus[k]}(\theta) \vdash)$ do not contain wffs annotated with k. So the hypothetical inconsistency of the set $\mathsf{wff}(f_{\setminus[k]}(\theta) \vdash)$ is due to the occurrence in θ of some wffs(s) annotated with numeral(s) different from k. Observe that the inconsistency of $\mathsf{wff}(f_{\setminus[k]}(\theta) \vdash)$ yields the inconsistency of the set $\mathsf{wff}(\theta \vdash)$. However, $\sigma \vdash$ is a constituent of Γ_{j+1} because a rule has been applied to Γ_j w.r.t. $\theta \vdash$ and a wff annotated with a numeral different from k. Thus, by Lemma 10.2, the set $\mathsf{wff}(\sigma \vdash)$ is inconsistent. Moreover, its inconsistency is due to the occurrence in σ of wffs annotated with numerals different from k. Therefore the set $\mathsf{wff}(f_{\setminus[k]}(\sigma) \vdash)$ is inconsistent. We arrive at a contradiction. This completes the proof of (★).

The sequence (10.14) is supposed to be a $\mathsf{MI}^{\mathsf{CPL}}$-proof, so, by Definition 10.11, for any $k \in \{1, \ldots, m\}$ there exists a constituent, say, $\rho \vdash$, of Γ_n such that, as $f_{\setminus[k]}(\rho \vdash)$ is either $\emptyset \vdash$ or is an open atomic sequent, the set $\mathsf{wff}(f_{\setminus[k]}(\rho) \vdash)$ is consistent. Thus, by (★) proven above, any term/seqsequent of (10.14) has a constituent, $\zeta \vdash$, such that

$$\mathsf{wff}(f_{\setminus[k]}(\zeta) \vdash)$$

is a consistent set of wffs. But the sequent $C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash$ is the only constituent of Γ_1. Hence $\mathsf{wff}(f_{\setminus[k]}(C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash)$ is a consistent set. As k was an arbitrary element of $\{1, \ldots, m\}$, by Corollary 10.2 it follows that X is a MI-set. □

Due to Theorem 10.1, in order to show that X is a MI-set it suffices to prove an ordered sequent $C^{[\overrightarrow{m}]}_{\overrightarrow{m}} \vdash$ for which the equation $X = \mathsf{wff}(C^{[\overrightarrow{m}]}_{\overrightarrow{m}})$ holds. Thus, for instance, the proofs of ordered sequents (10.5), (10.6), (10.7), (10.8), (10.9), (10.10), (10.11), (10.12), and (10.13) presented in section 10.3.3 demonstrate that the following are MI-sets, respectively:

$$\{p, \neg p\}$$

$$\{p \wedge \neg p\}$$

$$\{p, \neg(\neg p \rightarrow q)\}$$

$$\{p \vee q, \neg p, \neg q\}$$

$$\{p \rightarrow q, p, \neg q\}$$

$$\{p \wedge (\neg p \vee q), \neg q\}$$

$$\{p \rightarrow (q \rightarrow r), p \wedge q, \neg r\}$$

$$\{p \vee (q \vee r)), \neg(p \vee q), \neg(p \vee r)\}$$

$$\{p \rightarrow (q \rightarrow r), p \rightarrow q, \neg(p \rightarrow r)\}$$

10.4.3 Completeness w.r.t. MI-sets

The system $\mathsf{MI}^{\mathsf{CPL}}$ is complete w.r.t. MI-sets.

An auxiliary concept is needed.

Definition 10.12 ($\mathsf{MI}^{\mathsf{CPL}}$-transformation of a sequent)**.**
A $\mathsf{MI}^{\mathsf{CPL}}$-transformation of a sequent $\sigma \vdash$ is a finite sequence $\Gamma_1, \ldots, \Gamma_n$ of seqsequents such that:

1. $\Gamma_1 = \langle\langle\sigma \vdash\rangle\rangle$, *and*
2. Γ_{j+1} *results from* Γ_j *by a rule of* $\mathsf{MI}^{\mathsf{CPL}}$ *for* $1 \leqslant j < n$.

Note that while the concept of proof has been defined for ordered sequents only, this restriction is lifted in the case of $\mathsf{MI}^{\mathsf{CPL}}$-transformations.

Let us now prove:

Theorem 10.2 (Completeness w.r.t. MI-sets)**.** *If X is a MI-set, then any ordered sequent $\sigma \vdash$ such that $\mathsf{wff}(\sigma \vdash) = X$ is provable in $\mathsf{MI}^{\mathsf{CPL}}$.*

Proof. A moment's reflection on the rules of $\mathsf{MI}^{\mathsf{CPL}}$ reveals that for each ordered sequent $\sigma \vdash$ such that $\mathsf{wff}(\sigma \vdash)$ is an inconsistent set of wffs, there exist $\mathsf{MI}^{\mathsf{CPL}}$-transformations of the sequent which end with seqsequents whose constituents are closed atomic sequents only.

Assume that X is a MI-set, and that $C_{\vec{m}}^{[\vec{m}]} \vdash$ is an ordered sequent such that $\mathsf{wff}(C_{\vec{m}}^{[\vec{m}]} \vdash) = X$. Since X is a MI-set, $\mathsf{wff}(C_{\vec{m}}^{[\vec{m}]} \vdash)$ is an inconsistent set. Let

$$\Gamma_1, \ldots, \Gamma_n \tag{10.15}$$

be a $\mathsf{MI}^{\mathsf{CPL}}$-transformation of the sequent $C_{\vec{m}}^{[\vec{m}]} \vdash$ such that each constituent of Γ_n is a closed atomic sequent. Suppose that the transformation (10.15) is not a proof of $C_{\vec{m}}^{[\vec{m}]} \vdash$.

The transformation (10.15) can be depicted as:

$$\begin{aligned} &\Gamma_1 \Leftarrow \mathsf{R}_x^{[i_1]} \\ &\Gamma_2 \Leftarrow \mathsf{R}_x^{[i_2]} \\ &\quad \ldots \\ &\Gamma_{n-1} \Leftarrow \mathsf{R}_x^{[i_{n-1}]} \\ &\Gamma_n \end{aligned} \tag{10.16}$$

where '$\Leftarrow \mathsf{R}_x^{[i_j]}$' indicates that the rule applied to Γ_j acts upon a wff annotated with i_j (more precisely, upon an occurrence of such a wff in a sequent that belongs to Γ_j).

If the transformation (10.15) is not a proof, then, by Definition 10.11, there exists an index k, where $1 \leqslant k \leqslant m$, such that for each sequent $\theta \vdash$ which occurs in Γ_n, $f_{\setminus[k]}(\theta) \vdash$ is a closed atomic sequent.

Suppose that $m = 1$. Thus X is a singleton set, each rule involved in (10.16) acts upon a wff annotated with 1, and all the wffs which occur in Γ_n are annotated with 1. Hence $f_{\setminus[1]}(\theta) \vdash = \emptyset \vdash$ for any constituent $\theta \vdash$ of Γ_n. It follows that there is no constituent of Γ_n such that $f_{\setminus[1]}(\theta \vdash)$ is a closed atomic sequent. We arrive at a contradiction.

Suppose that $m > 1$. We proceed as follows. First, we remove from (10.16) each Γ_j which is associated with $\Leftarrow \mathsf{R}_x^{[k]}$, that is, we skip all the lines of (10.15) in which a rule acts upon a wff annotated with k. Let

$$\Gamma_1^*, \ldots, \Gamma_h^* \tag{10.17}$$

stand for the subsequence of (10.15) obtained from it in this way. Each Γ_j^*, where $1 \leqslant j \leqslant h$, is a sequence of sequents.

Let $\Gamma_j^* = \langle \xi_1 \vdash, \ldots, \xi_s \vdash \rangle$. We define Γ_j^{**} as:

$$\langle f_{\setminus[k]}(\xi_1) \vdash, \ldots, f_{\setminus[k]}(\xi_s) \vdash \rangle \tag{10.18}$$

Since $f_{\setminus[k]}(\xi_i) \vdash$ is a sequent for $1 \leqslant i \leqslant s$, (10.18) is a seqsequent. Then we consider the following sequence of seqsequents:

$$\Gamma_1^{**}, \ldots, \Gamma_h^{**} \tag{10.19}$$

Observe that wffs annotated with k do not occur in any constituent of any element/term of (10.19). Clearly, $f_{\setminus[k]}(C_{\vec{m}}^{[\vec{m}]}) \vdash$ is

$$C_1^{[1]}, \ldots, C_{k-1}^{[k-1]}, C_{k+1}^{[k+1]}, \ldots, C_m^{[m]} \vdash \tag{10.20}$$

It is easily seen that (10.19) is a $\mathsf{MI}^{\mathsf{CPL}}$-transformation of the sequent (10.20). On the other hand, each constituent of Γ_h^{**} is a closed atomic sequent and hence $\mathsf{wff}(\theta \vdash)$ is inconsistent for any constituent $\theta \vdash$ of Γ_h^{**}. Thus, by Lemma 10.3,

$$\{C_1, \ldots, C_{k-1}, C_{k+1}, \ldots, C_m\} \tag{10.21}$$

is an inconsistent set of wffs. But (10.21) is a proper subset of X. Therefore X is not a MI-set. We arrive at a contradiction. This completes the proof. □

10.4.4 Soundness and Completeness w.r.t. Strong Entailments

As we have shown, the calculus $\mathsf{MI}^{\mathsf{CPL}}$ is sound and complete w.r.t. MI-sets. Due to Theorem 6.5, the fact that $X, \neg Y$ is a MI-set guarantees that $X \Vdash\!\prec Y$ holds provided that $X \cap \neg Y = \emptyset$ is the case. So when we restrict ourselves to ordered sequents built in such a way that the fulfilment of the latter condition is secured, proofs of these ordered sequents can be viewed as demonstrations that strong mc-entailment hold in the cases considered.

Theorem 10.3 (Soundness w.r.t. strong mc-entailment).
Let $X = \{A_1, \ldots, A_n\}$ and $Y = \{B_1, \ldots, B_k\}$, where $n + k > 0$ and $A_i \neq \neg B_j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k$. If the ordered sequent:

$$A_1^{[1]}, \ldots, A_n^{[n]}, \neg B_1^{[n+1]}, \ldots, \neg B_k^{[n+k]} \vdash$$

is provable in $\mathsf{MI}^{\mathsf{CPL}}$, then $X \Vdash\!\prec Y$.

Proof. By Theorem 6.5 and Theorem 10.1. □

Example 10.35. $\mathsf{MI}^{\mathsf{CPL}}$-proofs of the following ordered sequents:

$$(p \wedge \neg p)^{[1]} \vdash \tag{10.6}$$

$$(p \vee q)^{[1]}, \neg p^{[2]}, \neg q^{[3]}, \vdash \tag{10.8}$$

$$p \vee (q \vee r))^{[1]}, \neg(p \vee q)^{[2]}, \neg(p \vee r)^{[3]} \vdash \tag{10.12}$$

presented in section 10.3.3 can be regarded as demonstrations that the following hold:

$$p \wedge \neg p \Vdash\!\prec \emptyset \tag{10.22}$$

$$p \vee q \Vdash\!\prec \{p, q\} \tag{10.23}$$

$$p \vee (q \vee r) \Vdash\!\prec \{p \vee q, p \vee r\} \tag{10.24}$$

The following is true as well:

Theorem 10.4 (Completeness w.r.t. strong mc-entailment)**.**
Let $X = \{A_1, \ldots, A_n\}$ *and* $Y = \{B_1, \ldots, B_k\}$*, where* $n + k > 0$ *and* $A_i \neq \neg B_j$ *for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, k$*. If* $X \Vdash\!\prec Y$*, then the ordered sequent:*

$$A_1^{[1]}, \ldots, A_n^{[n]}, \neg B_1^{[n+1]}, \ldots, \neg B_k^{[n+k]} \vdash$$

is provable in $\mathsf{MI}^{\mathsf{CPL}}$*.*

Proof. By Theorem 6.5 and Theorem 10.2. □

As for strong sc-entailment, one gets analogous results by applying Theorem 7.1 instead of Theorem 6.5.

Theorem 10.5 (Soundness w.r.t. strong sc-entailment)**.**
Let $X = \{A_1, \ldots, A_n\}$*, where* $n \geqslant 0$ *and* $A_i \neq \neg B$ *for* $i = 1, \ldots, n$*. If the ordered sequent:*

$$A_1^{[1]}, \ldots, A_n^{[n]}, \neg B^{[n+1]} \vdash$$

is provable in $\mathsf{MI}^{\mathsf{CPL}}$*, then* $X \models\!\prec B$*.*

Theorem 10.6 (Completeness w.r.t. strong sc-entailment)**.**
Let $X = \{A_1, \ldots, A_n\}$*, where* $n \geqslant 0$ *and* $A_i \neq \neg B$ *for* $i = 1, \ldots, n$*. If* $X \Vdash\!\prec Y$*, then the ordered sequent:*

$$A_1^{[1]}, \ldots, A_n^{[n]}, \neg B^{[n+1]} \vdash$$

is provable in $\mathsf{MI}^{\mathsf{CPL}}$*.*

Example 10.36. In section 10.3.3 we presented $\mathsf{MI}^{\mathsf{CPL}}$-proofs of the following ordered sequents:

$$p^{[1]}, \neg p^{[2]} \vdash \tag{10.5}$$

$$p^{[1]}, \neg(\neg p \to q)^{[2]} \vdash \tag{10.7}$$

$$(p \to q)^{[1]}, p^{[2]}, \neg q^{[3]} \vdash \tag{10.9}$$

$$(p \wedge (\neg p \vee q))^{[1]}, \neg q^{[2]} \vdash \tag{10.10}$$

$$(p \to (q \to r))^{[1]}], (p \wedge q)^{[2]}, \neg r^{[3]} \vdash \tag{10.11}$$

$$(p \to (q \to r))^{[1]}, (p \to q)^{[2]}, \neg(p \to r)^{[3]} \vdash \tag{10.13}$$

These proofs can be viewed as demonstrations that the following hold:

$$p \mathrel{|\!\!\!<} p \tag{10.25}$$

$$p \mathrel{|\!\!\!<} \neg p \to q \tag{10.26}$$

$$\{p \to q, p\} \mathrel{|\!\!\!<} q \tag{10.27}$$

$$p \wedge (\neg p \vee q) \mathrel{|\!\!\!<} q \tag{10.28}$$

$$\{p \to (q \to r), p \wedge q\} \mathrel{|\!\!\!<} r \tag{10.29}$$

$$\{p \to (q \to r), p \to q\} \mathrel{|\!\!\!<} p \to r \tag{10.30}$$

The proofs of the ordered sequents (10.8) and (10.12) can be *also* regarded as providing demonstrations of:

$$\{p \vee q, \neg p\} \mathrel{|\!\!\!<} q \tag{10.31}$$

$$\{p \vee (q \vee r), \neg(p \vee q)\} \mathrel{|\!\!\!<} p \vee r \tag{10.32}$$

10.5 $\mathsf{MI}^{\mathsf{CPL}}$ versus Tableaux Calculi and Erotetic Calculi

The primary rules of $\mathsf{MI}^{\mathsf{CPL}}$ transform wffs inside sequents analogously as Smullyan's tableaux rules do. It is possible to build a calculus of MI-sets in the "standard" tableau format, with rules defined as operating directly on (annotated) wffs, while occurrences of these wffs are nodes of respective trees. This would require adding an annotation mechanism and specifying new closing conditions. The advantage of the current format over the "standard" tableaux approach lies in its relative simplicity at the metatheoretical level.

The format of $\mathsf{MI}^{\mathsf{CPL}}$ is akin to that of the so-called erotetic calculi (cf., e.g., [57], [29]). Calculi of this kind exist for Classical Logic and a wide class of non-classical logics (cf. [29], [31], [67], [10]). The main difference between the format of $\mathsf{MI}^{\mathsf{CPL}}$ and that of erotetic calculi lies in the fact that rules of $\mathsf{MI}^{\mathsf{CPL}}$ operate on sequences of sequents, while rules of erotetic calculi act upon questions based on sequences of sequents. Moreover, annotations are exploited here in a new manner, and closing conditions of a proof are more demanding.

It is known that proofs written in the erotetic calculi format can be transformed into proofs in tableaux calculi (cf. [29]), sequent calculi (cf. [30], [29]) or even Hilbert-style calculi (cf. [18]). It is an open problem whether a similar effect shows up (and if yes, how) in the case of seqsequent calculi.

Chapter 11

Some Further Applications of the Formalism of $\mathsf{MI}^{\mathsf{CPL}}$

$\mathsf{MI}^{\mathsf{CPL}}$-proofs are devices by means of which MI-sets and strong entailments can be calculated. However, the formal apparatus of $\mathsf{MI}^{\mathsf{CPL}}$ enables accomplishments of other tasks as well.

11.1 Disproofs

The system $\mathsf{MI}^{\mathsf{CPL}}$ is useful not only in showing that something is a MI-set, but also in demonstrating that a set of wffs is inconsistent yet not minimally so. The latter can be achieved by providing a *disproof* of an ordered sequent which corresponds to the set of wffs under consideration.

Definition 11.1 ($\mathsf{MI}^{\mathsf{CPL}}$-disproof of an ordered sequent). *A finite sequence of seqsequents $\Gamma_1, \ldots, \Gamma_n$ is a $\mathsf{MI}^{\mathsf{CPL}}$-disproof of an ordered sequent $C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash$ iff*

1. $\Gamma_1 = \langle C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash \rangle$,
2. *Γ_{j+1} results from Γ_j by a rule of $\mathsf{MI}^{\mathsf{CPL}}$, where $1 \leqslant j < n$,*
3. *each constituent of Γ_n is a closed atomic sequent,*
4. *there exists $k \in \{1, \ldots, m\}$ such that for each constituent $\sigma \vdash$ of Γ_n, the sequent $f_{\setminus [k]}(\sigma) \vdash$ is closed.*

An ordered sequent is disprovable in $\mathsf{MI}^{\mathsf{CPL}}$ iff the sequent has a $\mathsf{MI}^{\mathsf{CPL}}$-disproof.

Observe that proofs and disproofs differ only with respect to their closing conditions. To be more precise, each sequence of seqsequents $\Gamma_1, \ldots, \Gamma_n$ satisfying the clauses (*1*), (*2*), and (*3*) of the definition of proof (i.e. Definition 10.11) and violating clause (*4*) of the definition is not a proof, but a disproof.

The following holds:

Theorem 11.1. *If there exists a* $\mathsf{MI}^{\mathsf{CPL}}$*-disproof of an ordered sequent* $C_{\vec{m}}^{[\vec{m}]} \vdash$*, then the set* $\mathsf{wff}(C_{\vec{m}}^{[\vec{m}]} \vdash)$ *is inconsistent, but is not a* MI*-set.*

Proof. Let

$$\Gamma_1', \ldots, \Gamma_n' \tag{11.1}$$

be an arbitrary but fixed disproof of $C_{\vec{m}}^{[\vec{m}]} \vdash$. Everything what has been said in the proof of Theorem 10.2 about the transformation (10.15), can be repeated with regard to the disproof (11.1) (of course, after replacing Γ_j with Γ_j' for $1 \leqslant j \leqslant n$). So $\mathsf{wff}(C_{\vec{m}}^{[\vec{m}]} \vdash)$ is not a MI-set. Yet, due to Definition 11.1 and Lemma 10.2, it is an inconsistent set. □

Example 11.37. Here is an example of a disproof of:

$$(p \to q)^{[1]}, p^{[2]}, \neg(p \vee q)^{[3]} \vdash$$

(clause (*4*) is satisfied with regard to numeral 1):

$$\begin{array}{cl}
(p \to q)^{[1]}, p^{[2]}, \neg(p \vee q)^{[3]} \vdash & \\
\hline
\neg p^{[1]}, p^{[2]}, \neg(p \vee q)^{[3]} \vdash;\ q^{[1]}, p^{[2]}, \neg(p \vee q)^{[3]} \vdash & \mathsf{R}_{\beta}^{[2]} \\
\hline
\neg p^{[1]}, p^{[2]}, \neg p^{[3]}, \neg q^{[3]} \vdash;\ q^{[1]}, p^{[2]}, \neg(p \vee q)^{[3]} \vdash & \mathsf{R}_{\alpha}^{[2]} \\
\hline
\neg p^{[1]}, p^{[2]}, \neg p^{[3]}, \neg q^{[3]} \vdash;\ q^{[1]}, p^{[2]}, \neg p^{[3]}, \neg q^{[3]} \vdash & \mathsf{R}_{\alpha}^{[2]}
\end{array}$$

Thus $\{p \to q, p, \neg(p \vee q)\}$, though inconsistent, is not a MI-set.

Finally, the following holds as well:

Theorem 11.2. *If X is a finite inconsistent set of wffs which is not a* MI*-set, then any ordered sequent $\sigma \vdash$ such that $\mathsf{wff}(\sigma \vdash) = X$ is disprovable in* $\mathsf{MI}^{\mathsf{CPL}}$.

Proof. Let

$$\Xi = \{\sigma \vdash : \sigma \vdash \textit{ is an ordered sequent such that } \mathsf{wff}(\sigma \vdash) = X\}$$

Let Λ_{Ξ} be the set of all $\mathsf{MI}^{\mathsf{CPL}}$-transformations of sequents in Ξ. As X is inconsistent, Λ_{Ξ} includes a non-empty subset Λ^*_{Ξ} of $\mathsf{MI}^{\mathsf{CPL}}$-transformations each of which ends with a seqsequent involving closed atomic sequent(s) only. As X is not a MI-set, by Theorem 10.1 no element of Ξ has a $\mathsf{MI}^{\mathsf{CPL}}$-proof. Therefore each transformation in Λ^*_{Ξ} violates the fourth clause of Definition 10.11. Hence Λ^*_{Ξ} comprises $\mathsf{MI}^{\mathsf{CPL}}$-disproofs of the sequents in Ξ. Thus any ordered sequent $\sigma \vdash$ such that $\mathsf{wff}(\sigma \vdash) = X$ is disprovable in $\mathsf{MI}^{\mathsf{CPL}}$. □

11.2 Strict Multiple-Conclusion Entailment

Let us now consider another subrelation of multiple-conclusion entailment, which we dub *strict multiple-conclusion entailment* or strict mc-entailment for short. We use $\Vert\!\triangleright$ as the symbol of this relation, and define it as follows:[63]

Definition 11.2 (Strict mc-entailment). *$X \Vert\!\triangleright Y$ iff*

1. *$X \Vert\!= Y$, and*
2. *for each $B \in Y$: $X \Vert\!\not= Y_{\ominus B}$.*

The difference between strict mc-entailment and strong mc-entailment lies in the absence of the second clause of the definition of strong mc-entailment in the definition of strict mc-entailment. In both cases the hypothetical truth of all the wffs in X warrants the existence of at least one true wff in Y, and the warranty disappears as Y decreases. However, in the case of strict mc-entailment, in contradistinction to strong mc-entailment, it is not required that the warranty disappears as X decreases. Thus $\Vert\!\prec$ is a subrelation of $\Vert\!\triangleright$.

Leaving aside the issue of the area of applicability of the concept of strict mc-entailment[64], let us only note that the formalism of $\mathsf{MI}^{\mathsf{CPL}}$ enables us to calculate strict mc-entailment from finite sets of wffs.

Clearly, the following hold:

[63] Recall that $Y_{\ominus B}$ equals $Y \setminus \{B\}$.

[64] However, some ideas suggest themselves. For instance, when Y is an at least two-element finite set of wffs, the disjunction connective is understood classically, and $X \Vert\!\triangleright Y$ holds, then $\bigvee Y$ may be called a *minimal disjunction* entailed by X (the concept of minimal disjunction plays an important role in adaptive logics; see, e.g. [4]). If, in turn, Y is a set of direct answers to a question Q, then $X \Vert\!\triangleright Y$ provides an explication of the intuitive notion: "a question Q arises from the set of declarative wffs X" (cf. [66], p. 114).

Corollary 11.1. *$X \Vert\!\triangleright Y$ iff the set $X, \neg Y$ is inconsistent, and for each $B \in Y$, the set $X, \neg(Y_{\ominus B})$ is consistent.*

Corollary 11.2. *If $X \Vert\!\triangleright Y$ and $Y \neq \emptyset$, then $X \Vert\!\not\models Z$ for any proper subset Z of Y.*

Let us now prove:

Theorem 11.3. *Let $X = \{A_1, \ldots, A_n\}$ and $Y = \{B_1, \ldots, B_k\}$, where $n \geqslant 0$ and $k > 0$. If*

$$\Gamma_1, \ldots, \Gamma_m \tag{$\boxplus$}$$

is a $\mathsf{MI}^{\mathsf{CPL}}$-transformation of the sequent:

$$A_1^{[1]}, \ldots, A_n^{[n]}, \neg B_1^{[n+1]}, \ldots, \neg B_k^{[n+k]} \vdash \tag{$\boxtimes$}$$

such that ($\boxplus$) fulfils the following conditions:

(c_1) *each constituent of Γ_m is a closed atomic sequent,*

(c_2) *for each $h \in \{n+1, \ldots, n+k\}$ there exists a constituent $\sigma \vdash$ of Γ_m such that the sequent $f_{\setminus[h]}(\sigma) \vdash$ is an open atomic sequent or is of the form $\emptyset \vdash$.*

then $X \Vert\!\triangleright Y$.

Proof. Condition (c_1) together with Lemma 10.3 ensure that $X \Vert\!\models Y$. By a reasoning analogous to that performed in the proof of Theorem 10.1 we get:

($\bigstar'$) *if Γ_{j+1} has a constituent, $\sigma \vdash$, such that the set*

$$\mathsf{wff}(f_{\setminus[h]}(\sigma) \vdash)$$

is consistent, then Γ_j has a constituent, θ, such that the set

$$\mathsf{wff}(f_{\setminus[h]}(\theta) \vdash)$$

is consistent, where $1 \leqslant j < m$, and $n+1 \leqslant h \leqslant n+k$.

Thus, due to the condition (c_2), for each $B \in Y$, the set $X, \neg(Y_{\ominus B})$ is consistent and hence $X \Vert\!\not\models Y_{\ominus B}$. Therefore $X \Vert\!\triangleright Y$ holds. □

Thus in order to demonstrate that a finite set of wffs X strictly mc-entails a non-empty and finite set of wffs Y is suffices to perform a $\mathsf{MI}^{\mathsf{CPL}}$-transformation of the corresponding sequent such that the transformation satisfies the conditions (c_1) and (c_2) of Theorem 11.3.

11.3 $\mathsf{MI}^{\mathsf{CPL}}$ and a Uniform Account of Proofs and Refutations

As we have remarked in section 3.4 of Chapter 3, in the case of CPL validity, inconsistency and contingency of wffs are expressible in terms of MI-sets. One can easily prove:

Corollary 11.3.

1. *A wff C is valid iff $\{\neg C\}$ is a* MI*-set.*
2. *A wff C is inconsistent iff $\{C\}$ is a* MI*-set.*
3. *A wff C is contingent iff $\{C, \neg C\}$ is a* MI*-set.*

Given that the system $\mathsf{MI}^{\mathsf{CPL}}$ is sound and complete w.r.t. MI-sets (cf. theorems 10.1 and 10.2), from Corollary 11.3 we get:[65]

Theorem 11.4.

1. *A wff C is valid iff the sequent $\neg C^{[1]} \vdash$ is provable in* $\mathsf{MI}^{\mathsf{CPL}}$.
2. *A wff C is inconsistent iff the sequent $C^{[1]} \vdash$ is provable in* $\mathsf{MI}^{\mathsf{CPL}}$.
3. *A wff C is contingent iff the sequent $C^{[1]}, \neg C^{[2]} \vdash$ is provable in* $\mathsf{MI}^{\mathsf{CPL}}$.

$\mathsf{MI}^{\mathsf{CPL}}$-proofs have been defined as pertaining to ordered sequents. However, taking into account the content of clause (*1*) of Theorem 11.4, it makes no harm to define the concept of $\mathsf{MI}^{\mathsf{CPL}}$-proof of a wff as follows:

Definition 11.3 ($\mathsf{MI}^{\mathsf{CPL}}$-proof of a wff)**.** *A $\mathsf{MI}^{\mathsf{CPL}}$-proof of a wff C is the $\mathsf{MI}^{\mathsf{CPL}}$-proof of the sequent $\neg C^{[1]} \vdash$.*

By analogy to what we have done in Chapter 3, we can also introduce the respective concepts of $\mathsf{MI}^{\mathsf{CPL}}$-refutations of wffs.

Definition 11.4 ($\mathsf{MI}^{\mathsf{CPL}}$-refutation^1 of a wff)**.** *A $\mathsf{MI}^{\mathsf{CPL}}$-refutation^1 of a wff C is the $\mathsf{MI}^{\mathsf{CPL}}$-proof of the sequent $C^{[1]} \vdash$.*

Definition 11.5 ($\mathsf{MI}^{\mathsf{CPL}}$-refutation^2 of a wff)**.** *A $\mathsf{MI}^{\mathsf{CPL}}$-refutation^2 of a wff C is the $\mathsf{MI}^{\mathsf{CPL}}$-proof of the sequent $C^{[1]}, \neg C^{[2]} \vdash$.*

Clearly, the following is true:

[65] Note that sequents $\neg C^{[1]} \vdash$, $C^{[1]} \vdash$, and $C^{[1]}, \neg C^{[2]} \vdash$ are ordered sequents in view of Definition 10.10.

Theorem 11.5.

1. *A wff C is valid iff C has a* $\mathsf{MI}^{\mathsf{CPL}}$*-proof.*
2. *A wff C is inconsistent iff C has a* $\mathsf{MI}^{\mathsf{CPL}}$*-refutation*^{1}.
3. *A wff C is contingent iff C has a* $\mathsf{MI}^{\mathsf{CPL}}$*-refutation*^{2}.

Thus the system $\mathsf{MI}^{\mathsf{CPL}}$ provides *also* an account of validity, inconsistency, and contingency of CPL-wffs, and does it in a uniform way by introducing a common proof-theoretic mechanism for proofs and refutations of CPL-wffs.

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Andrzej Wiśniewski — ur. 1958 r. Ukończył studia filozoficzne na UAM w Poznaniu, gdzie uzyskał też stopień doktora (1986) oraz doktora habilitowanego (1991). W 1999 r. otrzymał tytuł profesora. Aktualnie pracuje w Zakładzie Logiki i Kognitywistyki Wydziału Psychologii i Kognitywistyki UAM. Jest autorem kilku monografii oraz kilkudziesięciu artykułów naukowych, z których większość ukazała się w wydawnictwach i czasopismach o zasięgu międzynarodowym. Jest laureatem Nagrody Fundacji na Rzecz Nauki Polskiej (2019), przyznanej za opracowanie koncepcji inferencyjnej logiki pytań oraz minimalnej semantyki erotetycznej.

Książka Andrzeja Wiśniewskiego dostarcza pogłębionej analizy uogólnień pojęcia konsekwencji, w szczególności w wersji, gdzie jako przesłanki rozważane są rodziny zbiorów zdań, a jako konkluzje – zbiory zdań. Zaprezentowane są też rozwiązania teoriodowodowe, w tym system pozwalający na wyprowadzanie formuł logicznie ważnych (czyli prawd logicznych), ale również na odrzucanie formuł sprzecznych, a nawet, co jest praktycznie niespotykane w literaturze, odrzucanie formuł kontyngentnych. Przedstawiono także nowatorskie pojęcie silnego wynikania, które opiera się na wyjściowym wynikaniu, jednak z narzuconym dodatkowym wymogiem jego minimalności. Zaprezentowano trzy paradoksy dotyczące pojęcia wiedzy. Wprowadzono pojęcie epistemicznej dozwoloności, różne od epistemicznej możliwości, ale odmienne od kategorii tego, co znane. Zaproponowano też rodzaj wynikania, określonego jako transmisja epistemicznej dozwoloności.

Należy pokreślić mnogość przedstawionych idei, wysoki poziom zaawansowania technicznego wypracowanego aparatu, jego znaczenie formalne, ale też aplikacyjne.

dr hab. Marek Nasieniewski, prof. UMK

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